

APPLICATIONS OF MARKOV CHAINS TO BUSINESS PROBLEMS

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Abstract

We present characteristic examples of applications of finite Markov chains to Business problems, which can be solved - as most of the problems concerning applications of Markov chains in general do - by distinguishing between two types of such chains, the ergodic and the absorbing ones.

Περίληψη

Στο άρθρο αυτό παρουσιάζουμε χαρακτηριστικά παραδείγματα εφαρμογών των πεπερασμένων αλυσίδων Markov σε εμπορικά προβλήματα, τα οποία μπορούν να επιλυθούν – όπως συμβαίνει γενικά με τα περισσότερα προβλήματα, που αφορούν εφαρμογές των αλυσίδων Markov - κάνοντας διάκριση μεταξύ δύο τύπων τέτοιων αλυσίδων, των εργοδικών και των απορροφητικών.

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1. Introduction

One of the most interesting approaches for the nature and position of mathematics into the whole body of the human knowledge is the one that determines it as an activity whose target is to interpret and (or) describe in a satisfactory way the various phenomena of the real world, as well as the real situations of our everyday life. This approach, combined with the recent waves of social, economical, and technological changes and evolutions of our world, transubstantiates mathematics to an essential factor of the formation of the future of our society.

The above target is succeeded through the use of **mathematical models**, which are simplified representations of the corresponding real phenomena and situations, achieved through the use of mathematical terms and symbols, i.e. functions, equations, inequalities, etc. A mathematical model excels compared with the other types of models, e.g. simulation, analogical, iconic, etc (Taha, 1967), used for the same purposes, because it gives accurate and general solutions to the corresponding problems. It is of worth to notice at this point the words of Davis and Hersh (1981) about it: ‘The usefulness of a mathematical model is exactly its success to foresee, or (and) to imitate accurately the behaviour of the real world’.

A very important theory that offers ideal conditions for the study and mathematical modelling of a certain kind of phenomena depending upon random variables is the theory of Markov processes. Roughly speaking a **Markov chain** is a stochastic process that moves in a sequence of steps (phases) through a set of states and has ‘no memory’, i.e. the probability of entering a certain state in a certain step, although it is not necessarily independent of previous steps, depends at most on the state occupied in the previous step. This is known as the **Markov property**.

When the set of its states is a finite set, then we speak about a **finite Markov chain**. For special facts on such type of chains we refer freely to Kemeny & Snell (1976).

The basic concepts of Markov chains were introduced by A. Markov in 1907 on coding literary texts. Since that time the Markov chain theory was developed by a number of leading mathematicians such as A. Kolmogorov, W. Feller etc, but only from the 60’s the importance of this theory to the Natural, Social and most of the other Applied Sciences has been recognized, e.g. Suppes and Atkinson (1960), Kemeny & Snell (1963), Bartholomew (1973) etc.

Most of the problems concerning applications of Markov chains can be solved by distinguishing between two types of such chains, the ergodic and the absorbing ones.

The purpose of this paper is to present characteristic applications of finite Markov chains to Business problems.

2. The basic form of the Markov model - Ergodic Markov chains

The basic form of the Markov model is used to make short run forecasts for the evolution of various phenomena. On the contrary, through the equilibrium situation of an ergodic chain, one obtains a long run forecast for the evolution of the corresponding phenomenon. Towards this direction important is also the role of the mean first passage matrix of the ergodic chain. We may recall that a Markov chain is said to be an *ergodic chain*, if it is possible to go between any two states, not necessarily in one step.

In this section we shall illustrate all the above through the following two examples on business problems:

Problem 1: A company circulates for first time in the market a new product, say K. The market's research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week.

i) Find the market's share for K two weeks after its first circulation, provided that the market's conditions remain unchanged.

ii) Find the market's share for K in the long run, i.e. when the consumers' preferences will be stabilized.

iii) Find how many times on average a consumer buys for first time competitive products between two purchases of K and vice versa. Find also how many times the consumer buys on average K for first time before buying a competitive product and vice versa.

Solution: i) We form a Markov chain having the following two states: s_1 = the consumer buys K, and s_2 = the consumer buys another competitive product (obviously this process has the Markov property).

Denote by p_{ij} the transition probability from s_i to s_j , $i, j=1, 2$, then the matrix

$$A = [p_{ij}] = \begin{matrix} & \begin{matrix} s_1 & s_2 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \end{matrix} & \begin{bmatrix} 0,7 & 0,3 \\ 0,2 & 0,8 \end{bmatrix} \end{matrix}$$

is called the *transition matrix* of the chain. Since the transition from a state to some other state (including itself) is the certain event, we have that $p_{i1} + p_{i2} = 1$, for $i=1, 2$.

Consider also the row-matrix $P_k = [p_1^{(k)} \ p_2^{(k)}]$, known as the *probability vector* of the chain, which gives the probabilities $p_i^{(k)}$ for the chain to be in state i at step k , for $i=1, 2$ and $k=0, 1, 2, \dots$, where we obviously have that $p_1^{(k)} + p_2^{(k)} = 1$.

It is well known then that $P_{k+1} = P_k A$, therefore $P_k = P_0 A^k$.

In our case, since K circulates for first time in the market, we have that $P_0 = [0 \ 1]$, therefore $P_2 = P_0 A^2 = [0,3 \ 0,7]$.

Thus the market's share for K two weeks after its first circulation will be 30%.

ii) Obviously the above chain is an ergodic one. It is well known that, for an ergodic n -state chain, $n \geq 2$, as the number of its steps tends to infinity (long run), the chain tends to an equilibrium situation, where the probability vector P_k takes a constant price, say $P = [p_1 \ p_2 \ \dots \ p_n]$, called the *limiting probability vector* of the chain. Thus the equilibrium situation is characterized by the equality $P = PA$, with

$$p_1 + p_2 + \dots + p_n = 1.$$

For the present 2-state chain the above equality gives that $p_1 = 0,7p_1 + 0,2p_2$ and $p_2 = 0,3p_1 + 0,8p_2$, or equivalently $0,3p_1 - 0,2p_2 = 0$.

Solving the linear system of the above equation and of $p_1 + p_2 = 1$ one finds that $p_1 = 0,4$, i.e. the market's share for K in the long run will be 40%.

iii) For this case we must calculate the *mean first passage matrix* $M = [m_{ij}]$ of the chain, whose entry m_{ij} gives the mean number of steps needed to go from state s_i to s_j for first time, $i, j=1, 2$.

To do so we must calculate first the matrix $N = (I_2 - A + JP)^{-1}$, where I_2 is the 2×2 unitary matrix, A is the transition matrix of the chain, P is its limiting probability vector and $J = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A straightforward calculation

gives that

$$N = \begin{bmatrix} 0,4 & -0,15 \\ -0,1 & 0,35 \end{bmatrix}.$$

Then $M = (I_2 - N + EN_{dg})D$, where $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, N_{dg} is the 2×2 diagonal matrix whose diagonal elements are equal to the corresponding elements of N and $D = [d_{ij}]$ is the 2×2 diagonal matrix with $d_{ii} = \frac{1}{p_i}$, $i=1,2$.

Therefore a straightforward calculation gives that

$$M = \begin{bmatrix} \frac{10}{4} & \frac{5}{6} \\ \frac{5}{4} & \frac{10}{6} \end{bmatrix}.$$

Inspecting M we observe that a consumer buys on average $m_{11} = 2,5$ competitive products for first time between two purchases of K . Vice versa he (she) buys on average $m_{22} \approx 1,67$ times K for first time between two purchases of competitive products. Notice that the diagonal elements of M are (in all cases) equal to $m_{ii} = \frac{1}{p_i}$.

We also observe that a consumer buys on average $m_{12} \approx 0,83$ times K for first time before buying a competitive product, and vice versa he buys on average $m_{21} = 1,25$ times a competitive product for first time before buying K .

The next example concerns the application of a 3-state ergodic Markov chain to the production process of an industry:

Problem2: In an industry the production of a certain product is regulated according to the existing stock of it at the end of each day. Namely, if at the end of the day there exist unsatisfied orders or the stock is zero, then the production of the next day covers the unsatisfied orders plus two more metric units (m.u.). On the contrary, if there exists a non zero stock, there is no production for the next day.

We further know that the consumers' demand for the product is either 1 m.u. per day with probability 60%, or 2 m. u. per day with probability 40%.

Under the above conditions determine the probability to have unsatisfied orders in the long run and in how many days on average this will be repeated for first time.

Solution: Since the maximum product's demand is 2 m. u., the production of the factory at the first day must be 2 m. u. and therefore at the end of the day the stock is either zero or 1 m. u. .

In the former case the process is repeated in the same way. In the latter case the production of the next day is zero and therefore at the end of this day the stock is either zero (in this case the process is repeated in the same way), or there are unsatisfied orders of 1 m. u. .

In the latter case the production of the next day is 3 m. u., i.e. 1 m. u. to cover the unsatisfied orders of the previous day plus 2 m. u., and so on.

It becomes therefore evident that according to the above rhythm of production there are three possible situations at the end of each day: s_1 = unsatisfied orders of 1 m.u., s_2 = zero stock and s_3 = stock of 1 m.u. .

Evidently our problem can be described with an ergodic Markov chain having as states the above possible situations s_i , $i=1,2,3$.

Using the given data it is easy to observe that the transition matrix of the chain is

$$A = \begin{matrix} & \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 0 & 0,4 & 0,6 \\ 0 & 0,4 & 0,6 \\ 0,4 & 0,6 & 0 \end{bmatrix} \end{matrix}.$$

Let $P = [p_1 \ p_2 \ p_3]$ be the limiting probability vector of the chain, then the equation

$P = PA$ gives that $p_1 = 0,4p_3$, $p_2 = 0,4p_1 + 0,4p_2 + 0,6p_3$, and $p_3 = 0,6p_1 + 0,6p_2$.

Adding the first two of the above equations we find the third one. Solving the linear system of the first two equations and of $p_1 + p_2 + p_3 = 1$ one finds that $p_1 = 0,15$.

Therefore the probability to have unsatisfied orders in the long run is 15% and this will be repeated on average after $m_{11} = \frac{1}{0,15} \approx 6,67$ days for first time..

3. Absorbing Markov chains

A state of a chain is called absorbing if, once entered, it cannot be left. Further a Markov chain is called an **absorbing chain** if it has at least one absorbing state and if from every state it is possible to reach an absorbing state, not necessarily in one step.

In order to illustrate the use of absorbing Markov chains in business applications we consider the following problems:

Problem 3: The following process is applied in an agricultural cooperative for the collection and shelling of a product:

s_1 =collection, s_2 =sorting - refining, s_3 =packing and s_4 =shelling.

The past data show that, being in stage s_2 , there is a 20% probability that the quality of the product is not satisfactory. In this case the corresponding quantity is thrown away and a new collection is attempted.

(i) Find the probability for the completion of the above process in seven at most steps after its start, as well as the probabilities to be in each of the stages s_1 , s_2 and s_3 at the 7th step of the above process

(ii) Assuming further that the duration of each of the stages s_i , $i=1,2,3,4$, is on average 10, 4, 3 and 45 days respectively find the mean time needed for the completion of the whole process.

Solution: i) We form a 4-state chain having as states the corresponding stages s_i of the above process, $i=1,2,3,4$ (obviously the Markov property holds in this case too). Obviously this chain is an absorbing one, with s_4 its unique absorbing state. According to the given data it is straightforward to check that the transition matrix of the chain is

$$A = \begin{matrix} & \begin{matrix} s_1 & s_2 & s_3 & s_4 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0,2 & 0 & 0,8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

Further, since the above process starts always from s_1 , we have that $P_0 = [1 \ 0 \ 0 \ 0]$ and therefore $P_6 = P_0 A^6 = [0,008 \ 0 \ 0,032 \ 0,96]$.

Thus the probability for the completion of the process in seven at most steps is 96%. Actually in this case the process is completed either in 4 steps if no quantity of the product is thrown away, or in 6 steps, if the quantity of the product is thrown away once, because its quality was found to be unsatisfactory. We observe also that the probability to be at stage s_3 in the 7th step is 3,2% and this will happen if the corresponding quantities of the product are thrown away for two successive times. There is also a small probability 0,8% to be at stage s_1 in the 7th step. This will happen if the corresponding quantities of the product are thrown away for three successive times. Finally it is impossible to be at stage s_2 in the 7th step

ii) We bring the transition matrix A of the chain to its canonical form A^* by listing the absorbing state first and then we make a partition of A^* as follows:

$$A^* = \begin{matrix} & \begin{matrix} s_4 & s_1 & s_2 & s_3 \end{matrix} \\ \begin{matrix} s_4 \\ s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ - & - & - & - \\ 0 & 0 & 1 & 0 \\ 0 & 0,2 & 0 & 0,8 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}.$$

Then the **fundamental matrix** of the chain is given by $N = (I_3 - Q)^{-1}$, where

I_3 is the 3X3 unitary matrix and $Q = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0,2 & 0 & 0,8 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$ is the transition matrix of the

non absorbing states of the chain resulting from the above partition of A^* .
 A straightforward calculation gives that

$$N = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 1,25 & 1,25 & 1 \\ 0,25 & 1,25 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

It is well known that the entries of N give the mean number of times in each non absorbing state before absorption for each possible non absorbing starting state.

Therefore, since in our case the chain starts always from s_1 , the mean number of times in state s_1 is 1,25, in s_2 is also 1,25, and in s_3 is 1. Thus the mean time needed for the completion of the whole process is $1,25(10+4)+3+45 = 65,5$ days.-

When an absorbing Markov chain has more than one absorbing states, then the transition matrix R from the non absorbing to the absorbing states enables us to calculate the probabilities for the chain to reach a certain absorbing state, when it starts from a certain non absorbing state. The above situation is illustrated with the following example, which concerns the application of an absorbing Markov chain with two absorbing states to a “random – walk” problem:

Problem 4: A supermarket has three storehouses, say A_1 , A_2 and A_3 between two cities, say C_1 and C_2 , as it is shown in the below diagram:

C_1 ----- A_1 ----- A_2 ----- A_3 ----- C_2

For the delivery of the goods a truck starts its route every day from one of the storehouses and terminates it to one of the cities. The truck moves each time one place to the right or to the left with the same probability. Find the mean number of stops of the truck to each storehouse during its route and the probability to terminate its route to the city C_1 , when it starts it from storehouse A_2 .

Solution: We form a 5-state Markov chain having the following states:

s_1 (or s_5) = the truck arrives to the city C_1 (or C_2),

s_2 (or s_3 , or s_4) = the truck arrives to the storehouse A_1 (or A_2 , or A_3).

Obviously the above chain is an absorbing one and s_1 , s_5 are its absorbing states. Therefore the canonical form of its transition matrix is

$$A^* = \begin{matrix} & s_1 & s_5 & & s_2 & s_3 & s_4 \\ \begin{matrix} s_1 \\ s_5 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 0 \\ - & - & | & - & - & - \\ 0,5 & 0 & | & 0 & 0,5 & 0 \\ 0 & 0 & | & 0,5 & 0 & 0,5 \\ 0 & 0,5 & | & 0 & 0,5 & 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} I & | & 0 \\ - & | & - \\ R & | & Q \end{bmatrix}.$$

It is straightforward then to check that the fundamental matrix of the chain is

$$N = (I_3 - Q)^{-1} = \begin{matrix} & \begin{matrix} s_2 & s_3 & s_4 \end{matrix} \\ \begin{matrix} s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1,5 & 1 & 0,5 \\ 1 & 2 & 1 \\ 0,5 & 1 & 1,5 \end{bmatrix} \end{matrix}.$$

Thus, since the truck starts its route from the storehouse A_2 (state s_3), the mean number of its stops to the storehouse A_1 (state s_2) is 1, to the storehouse A_2 (state s_3) is 2 and to the storehouse A_3 (state s_4) is 1.

Further, the transition matrix from the non absorbing to the absorbing states resulting from the above partition of A^* is

$$R = \begin{matrix} & \begin{matrix} s_1 & s_5 \end{matrix} \\ \begin{matrix} s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0,5 & 0 \\ 0 & 0 \\ 0 & 0,5 \end{bmatrix} \end{matrix}.$$

It is well known that the entries of the matrix $B = NR$ give the probabilities for the chain to be absorbed in each absorbing state for each possible non absorbing starting state. It is straightforward to check that

$$B = \begin{matrix} & \begin{matrix} s_1 & s_2 \end{matrix} \\ \begin{matrix} s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0,75 & 0,75 \\ 0,5 & 0,5 \\ 0,25 & 0,25 \end{bmatrix} \end{matrix}.$$

Thus the probability for the truck to terminate its route to the city C_1 (state s_1), when it starts it from store A_2 (state s_3) is 50%. -

Our last example illustrates the fact that a great care is needed in certain cases in order to “translate” correctly the mathematical results of the Markov chain model in terms of the corresponding real situation.

Problem 5: In a college, where the duration of studies is four years, it has been estimated that for each student there is a 20% probability to be withdrawn due to unsatisfactory performance and a 30% probability to repeat the same year of studies. Find the probability for a student to graduate, the mean time needed for the graduation and the mean time of attendance in each year of studies.

Solution: The corresponding process can be described with the following states: s_i = attendance of the i – th year of studies, $i=1,2,3,4$, s_5 = withdrawal from the college and s_6 =graduation.

Obviously we can form an absorbing Markov chain, where s_5 and s_6 are its absorbing states. The canonical form of its transition matrix is

$$A^* = \begin{matrix} & \begin{matrix} s_5 & s_6 \end{matrix} & & \begin{matrix} s_1 & s_2 & s_3 & s_4 \end{matrix} \\ \begin{matrix} s_5 \\ s_6 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 0 & 0 \\ - & - & | & - & - & - & - \\ 0,2 & 0 & | & 0,3 & 0,5 & 0 & 0 \\ 0,2 & 0 & | & 0 & 0,3 & 0,5 & 0 \\ 0,2 & 0 & | & 0 & 0 & 0,3 & 0,5 \\ 0,2 & 0 & | & 0 & 0 & 0 & 0,3 \end{bmatrix} \end{matrix} = \begin{bmatrix} I & | & 0 \\ - & | & - \\ R & | & Q \end{bmatrix}$$

Further, using probably a proper PC mathematical package in order to make the necessary calculations quicker, it is straightforward to check that the fundamental matrix of the chain is

$$N = (I_4 - Q)^{-1} = \begin{matrix} & \begin{matrix} s_1 & s_2 & s_3 & s_4 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 1,429 & 1,02 & 0,729 & 0,521 \\ 0 & 1,429 & 1,02 & 0,729 \\ 0 & 0 & 1,429 & 1,02 \\ 0 & 0 & 0 & 1,429 \end{bmatrix} \end{matrix},$$

and therefore to find that

$$B=NR = \begin{matrix} & \begin{matrix} s_5 & s_6 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} & \begin{bmatrix} 0,74 & 0,261 \\ 0,636 & 0,365 \\ 0,49 & 0,51 \\ 0,286 & 0,715 \end{bmatrix} \end{matrix}.$$

Observe that from the fundamental matrix N of the chain, one finds that $n_{13}=0,729$ and $n_{14}=0,521$, i.e. for a first year student of the college the mean time of attendance to the third and fourth year of studies is less than one year! However this is not embarrassing, because there is always a possibility for a student to be withdrawn from the college due to unsatisfactory performance before entering the third, or fourth, year of studies.

Since $n_{11} = n_{22} = n_{33} = n_{44} = 1,429$, we find that the mean time of attendance of a student in each year of studies is 1,429 years, while the mean time needed for graduation is $1,429 \times 4 = 5,716$ years.

Further from B one finds that $b_{15} = 0,74$, i.e. the probability of a student to graduate is 74%.

4. Some other applications of Markov chains to real world problems

There are very many applications of Markov chains known in the literature for the solution of real world problems in almost every sector of the human activity, but a complete reference to all, or even to the most representative of them, is out of the scope of the present paper. Therefore we shall restrict here only to a brief presentation of some of our personal research results concerning applications of Markov chains to the sectors of management and economics and of mathematical education

In fact, in Voskoglou and Perdikaris (1991, 1993), and in Perdikaris and Voskoglou(1994) the problem-solving process is described through the introduction of an absorbing Markov chain to its main steps. Also in Voskoglou (1996 a) an ergodic chain is introduced for the study of the analogical problem-solving process in the classroom, while in Voskoglou (1996 b) an absorbing Markov chain is used for the description of the process of learning a subject matter by a group of students in the classroom

Further, in Voskoglou (1994) an absorbing Markov chain is introduced to the major steps through which one would proceed in order to effect the study of a real system (modelling process). The stochastic model obtained gives an important theoretical framework for the study of the modelling process. An alternative form of the above model is introduced in Voskoglou (2007) for the description of the mathematical modelling process in the classroom, when the teacher gives such kind of problems for solution to students. In this case it is assumed that after the completion of the solution process of each problem a new problem is given from the teacher to the class and therefore the process starts from the beginning again. Thus the resulting Markov chain is an ergodic one.

In Voskoglou (2000) an absorbing Markov chain is introduced to the main steps of the decision making process performed in order to choose the best among the existing solutions of a given problem, and examples are presented to illustrate the applicability of the constructed model to “real” decision making situations.

Finally in a recent paper (Voskoglou, 2009) the theory of Markov chains is used for the description of the Case -Based Reasoning (CBR) process, which is applied for the solution of a new problem in terms of already known from the past solutions of similar problems, and a measure is obtained for testing the efficiency of a CBR system.

5. Final conclusions

- The theory of Markov chains is a successful combination of Linear Algebra and Probability theory, which enables one to make short and long run forecasts for the evolution of various phenomena of the real world. In this paper we mainly presented characteristic applications of finite Markov chains to Business problems
- The short run forecasts are obtained, regardless the type of the chain, through the calculation of its transition matrix and the probability vector of the proper phase of the chain (basic form of the Markov model). On the contrary, the long run forecasts (equilibrium situation of the chain) are obtained, for the case of an ergodic chain only, by calculating its limiting probability vector. In this case the mean first passage matrix of the chain gives also some important information about the evolution of the corresponding phenomenon.
- In the case of absorbing Markov chains we proceed by forming the canonical form of its transition matrix and calculating the fundamental matrix of the chain, the entries of which give the mean number of times in each non absorbing state before absorption, for each possible non absorbing starting state. When an absorbing chain has more than one absorbing states, then the transition matrix from the non absorbing to the absorbing states enables us to calculate the probabilities for the chain to reach a certain absorbing state, when it starts from a certain non absorbing state.

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