

COMPARISON OF THE CLASSICAL METHODS WITH THE BOOTSTRAP. COMPUTER INTENSIVE TECHNIQUES IN ESTIMATING PARAMETERS OF THE BIVARIATE POISSON DISTRIBUTION

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Abstract

The bivariate Poisson distribution can be formed with the help of three independent univariate Poisson distributions. We generated values from a bivariate Poisson distribution via simulation and tried to estimate the covariance parameter which is of great importance since the correlation coefficient is linked to it. There are several methods to estimate the parameters of this distribution and hence the correlation coefficient. In this study we provide a comparison of various non parametric methods for estimating the true value of this coefficient. Techniques such as bootstrap-t, BCA, Fisher's approximation and more, are discussed along with their advantages and drawbacks. The Studentized Bootstrap presented a good coverage probability and a tolerable Confidence Interval and Mean Length.

Key words: Bivariate, Poisson, bootstrap, correlation, confidence intervals

1. Introduction

The bootstrap and jackknife are well-known resampling methodologies for obtaining nonparametric confidence intervals of or a parameter. In most statistical problems one needs an estimator of an unknown parameter of interest as well as some assessment of its variability. In many such problems the estimators are complicated functionals of the empirical distribution function and it is difficult to derive trustworthy analytical variance estimates for them. The bootstrap (Efron (1979, 1982, 1987)) and the jackknife (Miller (1974)) use straightforward but extensive computing to produce reliable indications of the variability of the estimator. For the justification of the bootstrap with regard to the foundation basis of the theoretical justification of bootstrap we have to say the following. The primary objective of this technique is to estimate the sampling distribution of a statistic. Essentially the bootstrap is a method that mimics the process of sampling from a population. Like one does in Monte Carlo simulations, by instead drawing samples from the observed sampling data. The tool of this mimic process is the wonderful Monte Carlo algorithm of Efron (1979). This process is explained properly in the book by Efron and Tibshirani (1993). Efron (1993) noted that the bootstrap confidence intervals are approximate yet better than the standard one. Nevertheless, they do not try to replace the theoretical ones. He also made clear that the bootstrap is not intended to be a substitute for precise parametric results, but rather a way to reasonably proceed when such results are unavailable.

“The bootstrap is in line with the general direction which Rao has imposed for the statistical discipline and is to extract all the possible information from the data” ([Rao, 1989](#)). One though, can pose the following question: why the bootstrap estimate of variance is converging to the true variance? The answer is the following: the strong law of large numbers implies that $\hat{S}_B \rightarrow \hat{S}$ almost surely as $B \rightarrow \infty$. What is the permissible necessary value of B? In this respect we use the reference of [Booth and Sarkar \(1998\)](#). These researchers suggested that a choice between 200 and 800 is satisfactory for the estimation of the standard error and thus the construction of confidence intervals. With the great power of computing today we can have a safe choice of B equal to 1000.

In this paper we have not used powerful bootstrap variations like the double bootstrap ([Tibshirani 1981](#) and Davison and Hinkley 1997) as well as the partial likelihood approach and the bootstrap calibration.

These methods have been proposed by Davison et. al. (1992), Beran (1987) and Loh (1987). The block bootstrap and the modified bootstrap were not implemented either.

The choice of the correlation coefficient for study was due to the fact that it has not been given much attention to this parameter. Most of the writers who dealt with it used the bivariate normal or lognormal distribution. Efron (1981) used a lot of bootstrap techniques to estimate the standard error of ρ in the bivariate normal distribution. He also saw that the non parametric delta method can be badly biased. Dolker M. et. al. (1982) pointed out some possible problems in the correlation coefficient for very small sample sizes. The probability of such discrepancies (same bivariate vectors) is very low.

Rasmussen J.L. (1987) worked on the estimation of the correlation coefficient for normal and nonnormal distributions and saw that the Pearson coefficient was robust to deviations from the normality assumption. Hall P. et. al. (1989), Frangos C.C. and Schucany W.R. (1990) and Lee W. and Rodgers J.L. (1998), studied the bivariate normal and lognormal distributions. In all cases continuous bivariate distributions were examined and mainly the normal distribution.

In this paper the estimator of interest is the correlation coefficient in two bivariate discrete distributions, the Poisson and the Negative Binomial. Section 2 of this paper describes the Normal method, the Basic method, Percentile method, the ABC method, the Studentized method and the accelerated bootstrap (BCa), using either the positive and negative jackknives to estimate the acceleration constant α . Section 3 describes the Classical method of constructing confidence intervals for the correlation coefficient. The comparison is accomplished by an extensive simulation study of the characteristics of interval estimators for correlation coefficient. Some characteristics of the selected methods are discussed in section 4 in terms of accuracy and transformation respecting property. The results are presented in section 5. The conclusions are that the Normal and the Basic methods do not work very well in these cases. The Studentized method does not work for low samples ($n=10$), while as the sample size increases it produces the best results. The BCa methods are very stable regardless of the sample size and the value of the correlation coefficient. The Fisher's Z transform performed also very well in all cases. What is common for all methods is the reduction of the average length as we move to higher values of the correlation coefficient and/or of the sample size.

2. Non parametric bootstrap confidence intervals

Let $X_1, X_2, \dots, X_n$, be n independent and identically distributed random variables from an unknown probability distribution $F_\theta(x)$. Let $\hat{F}$ be the usual empirical distribution function, having mass $1/n$ at each observed x_i . The essence of the bootstrap method is that one draws random samples $X_1^*, X_2^*, \dots, X_n^*$ with replacement from $\hat{F}$ and then calculates $\hat{\theta}^* = \hat{\theta}(X_1^*, \dots, X_n^*)$ as an estimate of θ . After B replications of this mechanism, one has an empirical distribution of $\hat{\theta}_b^*$ ($b=1, \dots, B$) values which serves as an estimate of the unknown sampling distribution of $\hat{\theta}$. Following Efron (1987), to construct nonparametric confidence intervals for θ by the percentile method, one uses the $100a$ and $100(1-a)$ percentiles of the bootstrap distribution of θ . The $1-2a$ central confidence interval for θ is given by

$$\theta \in [\hat{G}_B^{-1}(a), \hat{G}_B^{-1}(1-a)], \text{ where} \quad (2.1)$$

$$\hat{G}_B(t) = \frac{\#\{\hat{\theta}_b^* \leq t\}}{B} \quad (2.2)$$

is the estimated bootstrap distribution function.

The Normal method (or standard confidence interval) has the known form of

$$\theta \in [\hat{\theta} - Z^{(1-\alpha)} \cdot SE(\hat{\theta}), \hat{\theta} + Z^{(\alpha)} \cdot SE(\hat{\theta})], \quad (2.3)$$

where $SE(\hat{\theta})$ is a reasonable estimate of the standard error of $\hat{\theta}$ and $Z^{(\alpha)} = \Phi^{-1}(\alpha)$ obtained from the tables of standard normal distribution.

The bias-corrected method with acceleration constant α , (BC_a), introduced by Efron and Tibshirani (1986) and discussed in detail by Efron (1987), is a procedure for improved confidence intervals for

problems where there exists a monotone transformation g such that $\hat{\phi} = g(\hat{\theta})$ and $\phi = g(\theta)$ satisfying the approximation

$$\hat{\phi} \sim N(\phi - Z_0\sigma_\phi, \sigma_\phi^2), \text{ where } \sigma_\phi = 1 + \alpha\phi.$$

This yields the $1 - 2\alpha$ confidence interval

$$\theta \in \{G_B^{-1}(\Phi(Z[\alpha])), G_B^{-1}(\Phi(Z[1-\alpha]))\} \text{ where} \quad (2.4)$$

$$Z[\alpha] = Z_0 + \frac{Z_0 - Z_\alpha}{1 - \alpha(Z_0 - Z_\alpha)}.$$

Note that the BC intervals are BC, with $a = 0$ and they further reduce to PM when $Z_0 = 0$.

How does one find a ? Efron (1987) shows that for one-parameter families, $f_\theta(T)$, of sampling distributions of $T = \hat{\theta}$, a good approximation is

$$a = \frac{1}{6} SKEW_{\theta=\hat{\theta}}(i_\theta(T)), \quad (2.5)$$

where $SKEW_{\theta=\hat{\theta}}(i_\theta(T))$, is the skewness at $\theta = \hat{\theta}$ of the Fisher score function $i_\theta(T) = (\partial/\partial\theta) \ln f_\theta(T)$.

He also proposes that for data from an arbitrary distribution F_θ and $\theta = t(F_\theta)$ then a is reasonably well approximated by

$$\alpha = \frac{1}{6} \cdot \frac{\sum_{i=1}^n (I_i)^3}{\left\{ \sum_{i=1}^n (I_i)^2 \right\}^{3/2}}, \quad (2.6)$$

where I_i is the influence function of the functional t at the point x_i . Two different finite sample estimates of the influence function I_i are investigated here; the negative jackknife (I_-):

$$I_{i-} = (n-1)[(\hat{\theta}(x_1, \dots, x_n) - \hat{\theta}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n))], \quad (2.7)$$

and the positive jackknife (I_+):

$$I_{i+} = (n+1)[(\hat{\theta}(x_1, \dots, x_n, x_i) - \hat{\theta}(x_1, \dots, x_n))]. \quad (2.8)$$

The ABC method (stands for approximate bootstrap confidence intervals) is a method of approximating the BCa interval endpoints analytically, without using any Monte Carlo replications at all as Efron and Tibshirani describe (1993). It requires the resampling vector $P^* = (P_1^*, P_2^*, \dots, P_n^*)$ which consists of the proportions

$$P_i^* = N_i^* / n = \frac{\#\{x_j^* > x_i^*\}}{n}, \quad i=1, 2, \dots, n. \text{ Therefore the statistic e.g. } \hat{\theta}^* = \sum_{i=1}^n A_i^* \text{ is expressed as}$$

$$\hat{\theta}^* = \sum_{i=1}^n P_i^* A_i. \text{ The calculation of the confidence limits require the calculation of some empirical influence}$$

component and of the acceleration constant which is again calculated as $1/6$ times the standardized skewness of the empirical influence components.

Following Abramovitch and Singh (1985) and Loh (1987) we investigate a studentized statistic that is resampled to yield bootstrap confidence intervals for θ . The approximation of the distribution of $(\hat{\theta} - \theta) / SE(\hat{\theta})$, by the bootstrap distribution of the studentized pivotal quantity (SPQ) namely

$$t^* = (\hat{\theta}^* - \hat{\theta}) / SE(\hat{\theta}^*), \quad (2.9)$$

where $SE(\hat{\theta})$ is the square root of the estimated $Var(\hat{\theta})$. Denote by $G_s(t) = \#\{t_b^* < t\} / B$ the bootstrap distribution of the studentized quantity t^* . The bootstrap confidence interval has the form:

$$\theta \in [\hat{\theta} - G_s^{-1}(1-a)SE(\hat{\theta}), \hat{\theta} - G_s^{-1}(a)SE(\hat{\theta})] \quad (2.10)$$

The estimator for the standard error of the correlation coefficient was $(1 - \theta^2) / \sqrt{n-3}$.

The Basic method is a combination of the Percentile and the Studentized method. Instead of trying to find the empirical distribution of $\hat{\theta}_B^*$ ($b=1, \dots, B$), this method finds the empirical distribution of $\hat{\theta}_B^* - \hat{\theta}$. Note that this statistic is the same as in the studentized method with the only difference that the denominator (standard error of $\hat{\theta}$ is set equal to one). Then the confidence interval for the parameter θ is given by the formula:

$$\theta \in [2\hat{\theta} - \hat{G}^{-1}(1-a), 2\hat{\theta} - \hat{G}^{-1}(a)], \quad (2.11)$$

where $G_B^{-1}(t)$ is the same as in (2.2).

3. Classical confidence interval

The classical confidence interval for the correlation coefficient is extracted through the Z-transform, $\phi = 0.5 \ln\left(\frac{1+\rho}{1-\rho}\right) = \tanh^{-1}(\rho)$. The confidence intervals for z is

$$\phi \in [\hat{\phi} - Z_\alpha \sigma(\phi), \hat{\phi} + Z_\alpha \sigma(\phi)] = [L, U], \text{ where } \hat{\phi} = \tanh^{-1}(r) \text{ and } \sigma^2(\phi) = 1/(n-3).$$

It follows that the confidence intervals for the correlation coefficient are given by

$$\rho \in [(e^{2L} - 1)/(e^{2L} + 1), (e^{2U} - 1)/(e^{2U} + 1)]. \quad (3.1)$$

4. Characteristics of the above confidence intervals

The transformation respecting property is the characteristic that allows us to construct confidence intervals for a parameter and the transform the endpoints of the interval and end up with confidence intervals for a transform of the of the original parameter of interest. If for instance we have constructed a confidence interval for a parameter θ , then the interval for $\sqrt{\theta}$ will come easily by taking the square root of each of the endpoints of the initial confidence interval. The accuracy term refers to the rate of convergence, of the coverage probability, to the desired level of coverage. A central $(1-2\alpha)$ confidence interval is supposed have probability α of not covering the true value of θ from above and below. In the case of a sample this

probability is equal to $\alpha + \frac{c}{n}$ or $\alpha + \frac{c}{\sqrt{n}}$ for some constant c . In the first case the fraction goes to zero at rate

$1/n$, whereas in the second case it goes to zero at rate $1/\sqrt{n}$. We refer to the first case with the term second-order accuracy and with the term first-order accuracy to the second case.

The Normal (or Standard) method is known to be neither transformation respecting nor second-order accurate. The Percentile and the basic methods are transformation respecting, but not second-order accurate. The BCa method is both transformation respecting and second-order accurate. The Studentized method is

second order accurate but not transformation respecting. If the interest lies in the estimation of the correlation coefficient of the bivariate normal distribution then the Fisher's transform works quite well in this method. The problem arises in non normal populations like in our case. The drawback of not using a transform is that you can end up with an interval not satisfying the range restriction. We did not use any transform in the Studentized method and the result was obvious for small sized samples ($n=10$). The interval was larger than the set of permissible values for the correlation coefficient $([-1,1])$. This means that this method is not range-preserving. The ABC method is both second-order accurate and transformation respecting. One could also say that the Studentized method performs double bootstraps when the denominator is not known (in our case it is known), thus needs more computational effort. As already mentioned in the introduction the great power of computing these days, it is a serious problem no more. On the other hand the ABC method requires far less replications than its counterpart (BCa) needs. More information on the advantages and disadvantages of the various bootstrap methods can be found at Hall (1988) Efron and Tibshirani (1993) and DiCiccio and Efron (1996).

5. Simulations and results

A simulation study was performed as follows: 2000 independent-samples of size 10, 20, 50 and 100 were generated from each of two distributions; bivariate Poisson and bivariate Negative Binomial. The bivariate Poisson variates were generated according to the method described in the book of Morgan (1984). Three independent univariate Poisson variates are generated; say X_1 , X_2 and X_3 each with parameter λ_1, λ_2 and λ_3 respectively. Then $X=X_1+X_3$ and $Y=X_2+X_3$ follow the bivariate Poisson distribution with covariance parameter λ_3 . The parameters λ_1 and λ_2 were fixed at 0.5 and 1 respectively. What changed was the parameter of the covariance λ_3 . The probability mass function is given by this formula

$$P(X = x, Y = y) = \sum_{\delta=0}^{\min(x,y)} \frac{\lambda_1^{x-\delta} \lambda_2^{y-\delta} \lambda_3^{\delta}}{(x-\delta)!(y-\delta)!\delta!} \exp[-(\lambda_1 + \lambda_2 + \lambda_3)] \quad (5.1)$$

The correlation coefficient based upon this bivariate distribution is given by the following formula

$$\rho = \frac{\lambda_3}{\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}} \quad (5.2)$$

For the selected values of the correlation, 0.25, 0.5, 0.75 and 0.9, the values of the covariance were 0.2393, 0.7287, 2.2184 and 6.7123 respectively.

The bivariate Negative Binomial distribution can be generated with many ways, by the process of compounding a Poisson distribution as in the univariate case or with the help of two univariate Negative Binomial variates and one univariate Binomial variate. These methods are better described in the book of Kocherlakota S. and Kocherlakota K. (1992). For this paper the method of rejection sampling was used. For more information on the computer generation of the bivariate Negative Binomial distribution one can look at Loukas and Kemp (1986). The distribution function is given by the formula

$$P(X = x, Y = y) = \frac{(r+x+y-1)!}{(r-1)!x!y!} \cdot p_1^x p_2^y (1-p_1-p_2)^r \quad (5.3)$$

The correlation coefficient for this distribution is given by the following formula

$$\rho = \frac{\sqrt{p_1 p_2}}{\sqrt{(1-p_1)(1-p_2)}} \quad (5.4)$$

The values of the correlation coefficients were set equal to 0.25, 0.5 and 0.75. The confidence intervals for the correlation coefficient valued 0.9 in the bivariate Negative Binomial distribution were not

computed due to computational difficulties. The pair of parameters p_1 and p_2 took the values (0.1393, 0.2786), (0.2287, 0.4574) and (0.2898, 0.5796) respectively. The number of successes (r) was set equal to 5. The bootstrap confidence intervals with coverage probability $1-2\alpha=\gamma_n=0.95$ for the correlation coefficient were calculated by each of the eight methods: Normal, Basic, Percentile, ABC, Studentized, Classical, BCa(I_-) and BCa(I_+). Table 1 and 2 present the expected coverage and length of the confidence intervals for the Poisson and Negative Binomial cases respectively.

When the values of the correlation coefficient are less than or equal to 0.5 and the ample size is equal to 10 the average length exceeds unity. The Studentized method provides probably the best confidence intervals in general (amongst the ones compared) but with a price, the average length is the largest in all cases. The average length exceeded the length of the interval of possible values of the correlation coefficient $([-1,1])$. However as the sample size increases they seem to reach the nominal level of 0.95 faster and better than the other methods. This happens due to the fact that this method is not transformation-respecting. Since the pivotal quantity used here was not the same as the Fisher's this problem is natural to occur. The Normal and Basic methods did not work very well in general. As the correlation and the sample size increase they tend to provide better results. The average lengths are in accordance with the ones of the other methods, but this is not true for estimated coverage. The highest value the coverage was estimated to have was 92.6% for a large sample size ($n=100$) and high correlation ($\rho=0.9$). The graphs after the table 1 show in a better way the way the estimated coverage changes as the sample size increases. The lines seem to converge at a point a little lower than 0.95. The BCa and the Fisher's based intervals are the most stable in general. No matter of the value of the correlation and the sample size they tend to produce stable results in terms of coverage probabilities. The ABC method performs well for large sample sizes but in general is not to be preferred as it tends to underestimate the true coverage probability, but not more than the Normal and Basic methods. The Percentile method leads to pretty similar conclusions (Figure 1).

The same conclusions as before are to be drawn fro the case of the bivariate Negative Binomial distribution as well, perhaps a little better. The Normal and Basic methods perform the same as before and as the sample size increases from $n=20$ to $n=50$ the estimation of the coverage probability is more close to the nominal one but the average length is not reduced at the same amount as before. The Percentile method estimates the coverage to be more than 0.90 irrespectively of the sample size or the correlation value. The ABC method performance is about the same as before, perhaps a little worse. The stability of the BCa and the Fisher's methods are met in this case also. The Studentized confidence intervals are more conservative; they overestimate the nominal level of 0.95 for small samples but later approximate the desired level.

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Tables and Figures

Table 1

Estimates of the actual coverage, $\gamma_n = 0.95$ in percent (first line) and the expected length, (second line): Poisson distribution

Correlations	Sample sizes	Methods							
		Normal	Basic	Percentile	ABC	BCa(I_-)	BCa(I_+)	Studentized	Fisher's
$\rho=0.25$	n=10	85.4	76.6	89.6	90.45	92.8	93.6	98.4	94.2
		1.154	1.136	1.136	1.128	1.188	1.194	-*	1.126
	n=20	87.35	82.7	91.85	91.2	92.95	93.5	95.8	93.15
		0.826	0.819	0.819	0.794	0.826	0.829	1.064	0.829
	n=50	90.95	88.95	93.0	92.4	92.5	92.85	94.15	92.9
		0.534	0.533	0.533	0.526	0.534	0.534	0.584	0.534
	n=100	91.8	90.4	92.85	93.15	92.65	93.05	93.6	92.9
		0.386	0.386	0.386	0.383	0.387	0.386	0.404	0.386
$\rho=0.5$	n=10	84.35	75.4	90.1	87.74	92.35	93.8	96.55	92.75
		1.007	0.987	0.987	0.987	1.053	1.077	-	1.077
	n=20	88.55	82.5	92.3	91.4	93.45	94.2	95.85	93.4
		0.683	0.677	0.677	0.666	0.6927	0.701	0.875	0.701
	n=50	91.7	89.2	92.65	92.65	93.1	93.4	94.55	92.45
		0.44	0.44	0.44	0.439	0.444	0.446	0.482	0.446
	n=100	92.5	91.1	93.6	93.6	93.45	93.45	94.45	92.75
		0.316	0.316	0.316	0.316	0.317	0.317	0.33	0.317
$\rho=0.75$	n=10	86.85	76.0	90.8	87.995	93.4	94.15	96.45	93.35
		0.692	0.672	0.672	0.696	0.747	0.798	-	0.798
	n=20	87.75	81.45	90.95	90.8	92.05	92.7	93.7	92.75
		0.424	0.419	0.419	0.427	0.4407	0.455	0.54	0.455
	n=50	91.55	88.15	93.55	93.45	93.45	93.75	95.0	92.85
		0.263	0.263	0.263	0.267	0.269	0.272	0.285	0.272
	n=100	92.5	89.95	93.65	92.9	93.0	93.25	93.55	92.15
		0.187	0.187	0.187	0.19	0.19	0.191	0.195	0.191
$\rho=0.9$	n=10	87.7	73.3	89.4	84.2	91.6	92.5	95.25	93.15
		0.372	0.35	0.35	0.377	0.419	0.473	-	0.473
	n=20	88.45	81.7	91.75	90.2	91.65	92.1	93.5	93.8
		0.2	0.197	0.197	0.208	0.214	0.225	0.25	0.225
	n=50	91.5	86.05	93.7	93.65	93.5	93.65	94.4	93.4
		0.116	0.116	0.116	0.12	0.12	0.122	0.125	0.122
	n=100	92.6	90.7	93.55	93.85	94.4	94.35	94.15	92.9
		0.082	0.082	0.082	0.084	0.084	0.085	0.085	0.085

* Length larger than 2.

Figure 1. Poisson distribution

○=Normal, ■=Fisher, □=Percentile, ◇=ABC, ●=BCA(I₊), ▽=BCa(I₋), ▲=Studentized, △=Basic.

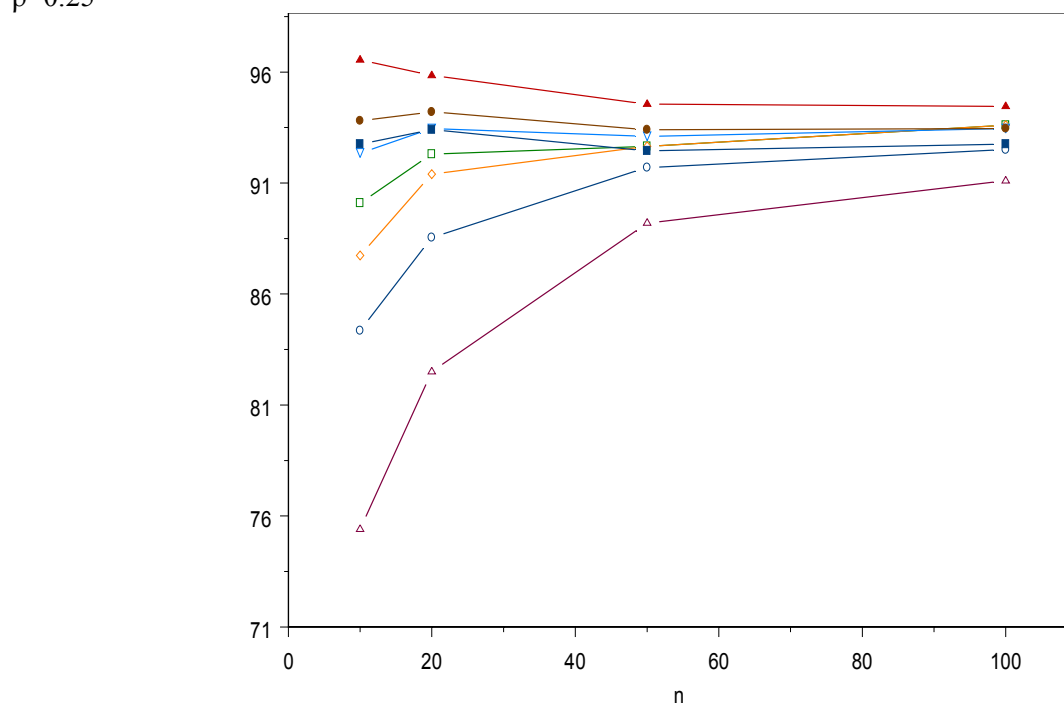
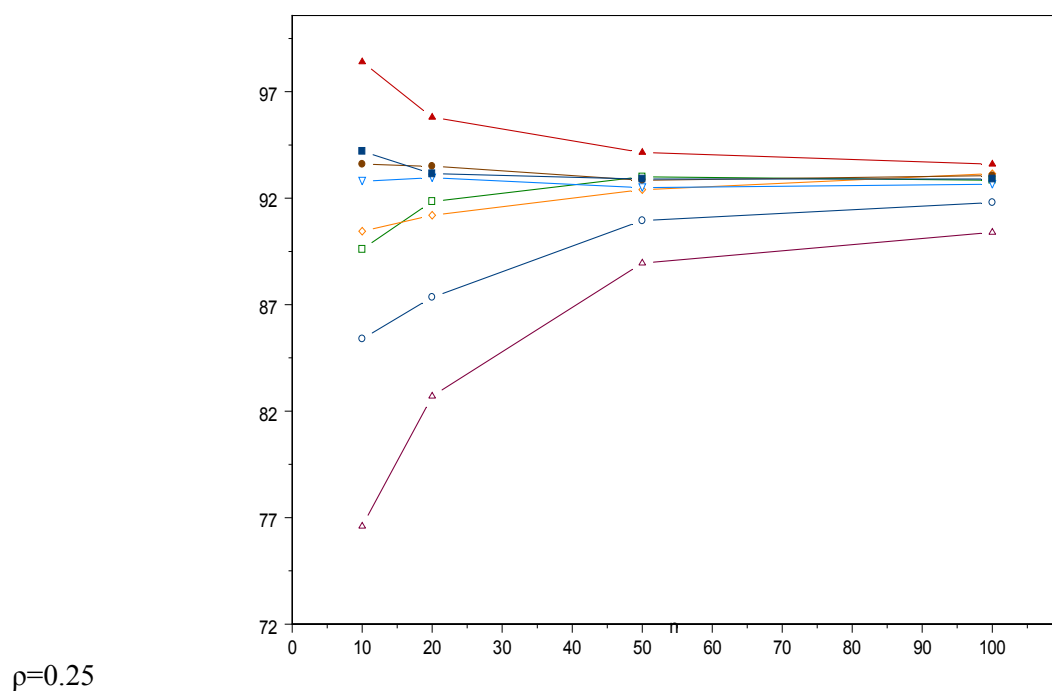


Table 2

Estimates of the actual coverage, $\gamma_n = 0.95$ in percent (first line) and the expected length, (second line):
Negative Binomial distribution

Correlations	Sample sizes	Methods							
		Normal	Basic	Percentile	ABC	BCa(I_-)	BCa(I_+)	Studentized	Fisher's
$\rho=0.25$	n=10	86.0	78.0	92.0	88.45	93.4	94.4	98.6	94.6
		1.19	1.173	1.173	1.151	1.209	1.218	-	1.218
	n=20	86.8	82.7	90.9	89.7	91.8	92.55	94.05	92.9
		0.824	0.817	0.817	0.787	0.825	0.826	1.05	0.826
	n=50	90.35	88.15	92.55	91.45	91.9	92.3	93.4	92.5
		0.535	0.533	0.533	0.525	0.535	0.534	0.583	0.534
	n=100	90.3	89.35	91.2	90.9	90.9	91.0	91.1	91.5
		0.387	0.386	0.386	0.384	0.387	0.387	0.404	0.387
	n=10	87.2	77.9	91.4	88.05	93.45	94.35	98.05	93.75
		1.032	1.012	1.012	1.004	1.077	1.098	-	1.098
$\rho=0.5$	n=20	88.9	83.9	91.7	89.85	92.0	92.85	94.65	93.5
		0.699	0.694	0.694	0.674	0.705	0.713	0.883	0.713
	n=50	92.55	90.25	94.4	93.25	93.25	93.8	95.05	93.7
		0.444	0.443	0.443	0.438	0.444	0.446	0.483	0.446
	n=100	93.3	91.8	93.05	92.7	92.45	92.7	93.7	92.55
		0.32	0.32	0.32	0.318	0.321	0.321	0.334	0.321
	n=10	87.95	76.0	91.45	86.2	92.55	93.85	96.75	92.9
		0.707	0.686	0.686	0.699	0.769	0.811	1.497	0.811
	n=20	88.9	82.15	91.95	90.0	92.25	92.55	94.2	92.35
		0.435	0.43	0.43	0.427	0.447	0.46	0.542	0.46
$\rho=0.75$	n=50	93.35	90.35	94.9	93.8	93.85	94.5	94.75	94.4
		0.264	0.263	0.263	0.264	0.267	0.269	0.284	0.269
	n=100	93.7	92.1	94.8	93.45	93.45	93.55	94.4	93.5
		0.187	0.187	0.187	0.187	0.188	0.188	0.194	0.189