

A THRESHOLD MODEL FOR MONETARY POLICY RULES FOR OPEN ECONOMIES

Thanassis Kazanas

Athens University of Economics and Business, Department of Economics
76 Patission Str., 104 34, Athens, Greece
Email: tkazanas@aueb.gr

Elias Tzavalis

Athens University of Economics and Business, Department of Economics
76 Patission Str., 104 34, Athens, Greece
Email: etzavalis@aueb.gr

Abstract

This paper estimates monetary policy reaction functions that allow for open economy. The famous Taylor rule is referred to a closed economy in which the interest rate depends on the inflation and output deviations from their target values. In an open economy the interest rate depends not only on the inflation and output deviations but also on the percentage change of the real exchange rate or the nominal interest rate of a foreign country. We consider a threshold type nonlinear monetary policy model that allows the existence of two policy regimes according to whether the output gap or the level of the unemployment rate is above or below a threshold value. We estimate the threshold parameter through the method of maximum likelihood. The results reveal asymmetries in the policy reaction functions which are associated with large output gap or high level of unemployment rate.

JEL Classification: E52, C13, C30

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1. Introduction

There is a growing number of simple monetary policy rules that the central banks may follow. The pioneering work of John Taylor who implemented a simple interest rate rule for the U.S. (Taylor 1993) has challenged the policy makers' interest. According to this rule the interest rates are adjusted in response to real output and inflation deviations from their target values. The literature that stems from this fact contains simple backward and forward looking rules, rules that allow for interest rate smoothing or not and also research that compares rules responding to alternative intermediate and final targets. Many economists advocate that the inflation targeting should be implemented through a simple Taylor rule. Ball (1997) and Svensson (1997) use optimal policy rules that are versions of inflation targets and Taylor rules.

These simple policy rules refer to a closed economy. For the case of an open economy, an economy where the exchange rate channel of monetary policy plays an important role in the transmission mechanism, the simple Taylor rule has to be modified. The model proposed by Ball (1999) assumes that the optimal policy instrument is a Monetary Condition Index based on both the interest rate and the exchange rate. He also replaces inflation with "long-run inflation", a variable that filters out the transitory effects of exchange rate fluctuations. Batini et al (2001) specify and evaluate a set of simple monetary policy rules that are suitable for a small open economy like the United Kingdom. The set of these simple rules also includes the simple Taylor rule and the rule proposed by Ball (1999). They come to the conclusion that an inflation forecast base rule, a rule that reacts to deviations of expected inflation from target, is a good simple rule. Adding in this setup contemporaneous and lagged levels of the real exchange rate the improvement is marginal. But this rule, with or without exchange rate adjustment, seems to be robust to different shocks in contrast to Taylor or Ball's rule.

Svensson (2000) uses a model with forward looking agents and explicit microfoundations regarding a policy rule where the short term nominal interest rate reacts to inflation, to the deviation of the real GDP from potential and to contemporaneous and lagged values of the real exchange rate. His simulations show a deterioration of the output performance. Taylor (1999) examines an analogous monetary policy rule to that of Ball and Svensson for the European Central Bank. His simulations lead to the conclusion that there are small performance improvements from a policy rule that reacts to the exchange rate or that the performance may be worse. McCallum and Nelson (2001) propose an alternative model to that of the New Open Economy Macroeconomics models. They construct a model by treating imports not as finished consumer

goods but rather as raw material inputs to the home country productive process. The calibration of their small open economy model leads to the fact that this model performs better than the standard New Open Economy Macroeconomics models in terms of matching the dynamic behavior of nominal variables.

Clarida, Gali and Gertler (1998) estimate monetary policy reaction functions for two sets of countries: the G3 (Germany, Japan and the U.S.) and the E3 (U.K., France and Italy). They use a forward looking version of the standard Taylor rule for a closed economy where the nominal interest rate reacts to the expected future inflation deviation from target and the contemporaneous output gap. They also estimate alternative specifications allowing for additional variables such as the real exchange rate or a foreign interest rate. This is done in order to be examined whether the home country takes the foreign country's monetary policy as external constraint. They regard as a foreign country the U.S. and estimate reaction functions for the Bundesbank and the Bank of Japan. Their estimates reveal a significant but quite small reaction of the interest rate to the exchange rate or the foreign interest rate. For the case of the European countries Germany is regarded as the foreign country and the results show a significant and quite high response of the interest rate to the German interest rate.

Kazanas, Philippopoulos and Tzavalis (2007, 2008) estimate backward and forward looking threshold reaction functions for some major central banks and their results reveal certain nonlinearities that are associated with the output gap or the unemployment rate. In this paper we extend this specification trying to investigate whether or not nonlinearities arise in the monetary reaction function of a central bank when its monetary policy is constrained by a foreign country. This research considers nonlinearity in the central bank's reaction function assuming that the central bank sets first its interest rate according to a threshold value of the output gap. It also examines if the unemployment rate plays a significant role as another threshold variable. The goal of the paper is to examine if evidence of nonlinearity in the central bank's reaction function is due to certain events which are associated with the cycle of the economy. We estimate the threshold variable of the bank of Japan regarding that its monetary policy is constrained by the Fed and assuming that the bank of Japan conducts its monetary policy through choosing as instrument the nominal interest rate.

The plan of the paper is as follows. Section 2 presents the baseline features of the nonlinear monetary policy rules. Section 3 presents the estimation procedure of our threshold type models and the corresponding results. Section 4 concludes the paper.

2. Nonlinear Monetary Reaction Function

The linear monetary policy rules are based on the assumptions that the central bank has a quadratic loss function and the Phillips curve is linear. The monetary policy of the Fed in the U.S., as well as in many countries, has been described very well according to this type of policy rule. The framework that we develop considers a central bank that has at least some degree of autonomy over its monetary policy. For the major central banks the main operating instrument of monetary policy is a short term interest rate. Typically, the instrument is an interbank lending rate for overnight loans. The empirical reaction functions that are estimated characterize the way that the central bank chooses the level of the short term interest rate from period to period. The class of these policy rules assumes the existence of temporary nominal wage and price rigidities. With these nominal rigidities the monetary policy affects real activity in the short run. A central bank can effectively vary the real interest rate and the real exchange rate by varying the nominal interest rate. In stating the policy reaction function we assume that within operating period the central bank has a target for the nominal interest rate, i_t^* , that is based on the state of the economy. This target depends on lagged values of the inflation deviation from its target, the output gap and the percentage change of the real exchange rate. So, we have the following backward looking policy rule:

$$i_t^* = a + \beta(\pi_{t-1} - \pi^*) + \gamma \tilde{y}_{t-1} + \delta z_{t-1} \quad (2.1)$$

where π_t is the percentage change in the price level between periods t and $t-1$ (expressed in annual rates), π^* is the inflation target and $\tilde{y}_t$ is a measure of the output gap in period t . $\tilde{y}_t$ is defined as the percentage change between actual GDP and the corresponding target (e.g. potential GDP). The variable z_t includes the percentage change of the real exchange rate. The parameter a is the long run equilibrium nominal interest rate when inflation and output are at their target values ($a = r^* + \pi^*$) with r^* being the long run equilibrium real interest rate.

The monetary policy rule given by (2.1) is restrictive if someone wants to describe the actual changes in the nominal interest rate. One main reason for this is the fact that our form does not take into account the central bank's tendency to smooth changes in interest rates but assumes an immediate adjustment of the interest rate to its target value. This central banks' tendency for smoothing stems from various reasons such as the fear of disrupting capital markets, the loss of credibility from sudden large policy reversals, the need for consensus building to support a policy change. Moreover, the central bank may regard the interest rate smoothing as a learning device due to imperfect information. All these factors that mentioned are indeed very difficult to be captured. For this reason we assume following Clarida et al. (1998) that the actual rate adjusts partially to its target according to:

$$i_t = (1 - \rho)i_t^* + \rho i_{t-1} + \varepsilon_t \quad (2.2)$$

with the parameter $\rho \in [0, 1]$ capturing the degree of smoothness and ε_t to be an i.i.d. exogenous shock with zero mean. Under this partial adjustment behavior the central bank wishes to adjust its instrument in order to eliminate only a fraction $(1 - \rho)$ of the gap between its current target level and some linear combination of its past values. Substituting (2.1) into (2.2) gives the following model for the backward looking monetary rule:

$$i_t = (1 - \rho) \left[a + \beta (\pi_{t-1} - \pi^*) + \gamma \tilde{y}_{t-1} + \delta z_{t-1} \right] + \rho i_{t-1} + \varepsilon_t \quad (2.3)$$

The above monetary policy rule assumes that the monetary authorities respond symmetrically to the variables of interest. But there is strong evidence that asymmetrical monetary policy rules may be attributed to asymmetrical central bank preferences or nonlinear structure of the economy. For that reason we adopt a monetary policy rule that will depend on whether a threshold variable is lesser or greater than a specific value. This rule takes into consideration that there are two different regimes governed by a threshold variable. For a value of the threshold variable less than or equal to the threshold parameter the central bank will follow the first regime while for a threshold variable value greater than the threshold parameter the central bank will follow the second regime. That is, the monetary policy rule is specified as

$$i_t = \begin{cases} (1 - \rho) \left[a + \beta (\pi_{t-1} - \pi^*) + \gamma \tilde{y}_{t-1} + \delta z_{t-1} \right] + \rho i_{t-1} + \varepsilon_{1t}, & \text{if } q_{t-1} \leq \bar{q} \\ (1 - \rho) \left[a + \beta' (\pi_{t-1} - \pi^*) + \gamma' \tilde{y}_{t-1} + \delta' z_{t-1} \right] + \rho i_{t-1} + \varepsilon_{2t}, & \text{if } q_{t-1} > \bar{q} \end{cases} \quad (2.6)$$

where $\bar{q}$ is the threshold parameter and q denotes the threshold variable. The innovation terms are assumed to be distributed as $\varepsilon_{1t} \sim i.i.d.(0, \sigma^2)$ and $\varepsilon_{2t} \sim i.i.d.(0, \sigma'^2)$. In the above specification, note that only the coefficients of inflation deviation from its target, the output gap and the real exchange rate are assumed to change their values between the two regimes. This means that the long run nominal interest rate a and the smoothing parameter ρ are constant across the two regimes. The above nonlinear model of monetary policy rule assumes that the threshold parameter is unknown. As threshold variables we will consider the first lag of the output gap and the first lag of the unemployment rate. The nonlinear model (2.5) can be employed to investigate how the central bank changes the magnitude of its reaction to the output gap and inflation deviations during recessions and expansions.

3. Empirical Results

In this section we estimate the policy reaction functions for the Bank of Japan assuming that its monetary policy is constrained by the Federal Reserve Bank of the U.S. The way that these central banks conduct their monetary policy has changed since the late of 70's. After nearly a decade of high inflation the inflation control became a major concern for the above central banks. We report our data set and after that we describe the estimation procedure to obtain our results.

3.1. Data

We use quarterly observations from 1972:I to 2006:IV. The interest rate is the respective interbank lending rate: the "Call-Money" rate for Japan and the Federal Funds rate for the U.S. expressed at annual rates in the first month of each quarter. For both countries the inflation is measured as the annualized rate of the GDP

deflator defined as $P_t = \frac{\text{nominal } GDP_t}{\text{real } GDP_t} \cdot 100$. More specifically, inflation is calculated as

$\pi_t = (P_t - P_{t-4}) \cdot 100$. As a measure of the output gap we employ the percentage change between real GDP

and potential real GDP $\left(\tilde{y}_t = \frac{\text{real } GDP_t - \text{real potential } GDP_t}{\text{real potential } GDP_t} \cdot 100 \right)$. The real GDP and the real potential

GDP are expressed in annual rates with the year 2000 being the base year. The unemployment rate is the seasonally adjusted civilian unemployment rate including persons 16 years of age and older at the last month of each quarter. We also use the level of the nominal yen/dollar exchange rate, e_t to construct the annualized percentage change in the real exchange rate, ϵ_t , as follows:

$$\% \text{ change in } e_t = \% \text{ change in } \epsilon_t + (\pi_t^{JP} - \pi_t^{US})$$

All these series were downloaded from the OECD database for the case of Japan and from the Federal Reserve Bank of St. Louis for the U.S.

3.2. Estimation of the baseline model

Before we estimate the baseline model, we give some descriptive statistics for our variables. These statistics are given in Table 1. The average output gap is negative for both countries, being near to zero for the case of Japan. U.S. has higher interest rates, inflation and unemployment rate than Japan.

Our baseline model is:

$$i_t = c_1 + c_2 \pi_{t-1} + c_3 \tilde{y}_{t-1} + c_4 z_{t-1} + c_5 i_{t-1} + \varepsilon_t \quad (3.1)$$

where $c_1 = (1 - \rho)(a - \beta\pi^*)$, $c_2 = (1 - \rho)\beta$, $c_3 = (1 - \rho)\gamma$, $c_4 = (1 - \rho)\delta$, $c_5 = \rho$. We see that our model can't identify a or π^* through the term $c_1 = (1 - \rho)(a - \beta\pi^*)$. For this reason, we have to predetermine the long run nominal interest rate a or the inflation target π^* in order the model (3.1) to be identified. We choose to predetermine a so as to be able to estimate the inflation target π^* . We take as the long run nominal interest rate the sample mean of the nominal interest rate i_t . After that we are able to estimate π^* jointly with the parameters β , γ , δ and ρ .

Using the augmented Dickey-Fuller test for unit root we have tested for the stationarity of the nominal interest rate, inflation, output gap series and exchange rate series. Our results give evidence for stationarity for all the variables employed in our analysis.

Table 2 reports estimates for the baseline reaction function parameters for the bank of Japan. In parenthesis we report the corrected test statistics for heteroskedasticity and autocorrelation. The results indicate an accommodative policy regarding the inflation deviation, as the coefficient β is below the unity despite the strong smoothing process. The coefficient of the output gap γ is statistically significant at the significance level of 10% and is greater than zero while the coefficient of the percentage change of the exchange rate is not statistically significant at the conventional levels indicating that the monetary authorities do not consider the fluctuations in the real exchange rate as a major problem. The estimation of the inflation target is $\pi^* = 2.698$.

In order to check whether the coefficients are stable in the whole period, or not we first apply the CUSUM test which is based on cumulative sums of squared residuals. According to this test we reject the null hypothesis that the coefficients are stable at a significance level of 5%. This test statistic finds structural breaks for the monetary rule in the mid 80's and 90's.

As there may be more than one change in the coefficients of the baseline model we next carry out Bai's and Perron (2003) tests for multiple structural changes. These tests allow us to estimate a pure or a partial structural change model with an unknown number of breaks. We carry out these tests in both, a pure structural and a partial structural model. In the partial structural model we allow the inflation gap, the output gap and the percentage change in the exchange rate coefficients to change among regimes. Also we allow the error terms in each segment to have different variance. In order to estimate the number of breaks, we use the Bayesian Information Criterion (BIC) and the method suggested by Bai and Perron which is based on the sequential application of the $\sup F_T(l+1/l)$ test using the sequential estimates of the breaks. The results are

presented in Table 3. These find a number of structural changes in the coefficients at the start of 80's and 90's. Note that these breaks are very close to those found by the CUSUM test.

3.3. Estimation of the nonlinear model

The nonlinear model (2.6) can be employed to investigate how the central bank changes the magnitude of its reaction to the output gap, to inflation deviations and the percentage change of the exchange rate during recessions and expansions. Regarding the unemployment rate as a threshold variable the central bank reactions may be different between the low and high unemployment rate regimes. Our estimate of the threshold model assumes that the threshold value $\bar{q}$ is unknown. In order to estimate the nonlinear model we construct the following joint log likelihood function*:

$$L(\theta) = \left(-\frac{T_1}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \varepsilon_1' \varepsilon_1 \right) + \left(-\frac{T_2}{2} \log(2\pi\sigma'^2) - \frac{1}{2\sigma'^2} \varepsilon_2' \varepsilon_2 \right) \quad (3.2)$$

where $\theta = (a, \beta, \gamma, \delta, \rho, \beta', \gamma', \delta', \sigma^2, \sigma'^2)'$. This log likelihood has to be maximized with respect to the population parameters.

To estimate the threshold parameter we follow Hansen's (1997, 2000) approach. We first sort our threshold variable in an ascending order and then the other variables according to the threshold variable. So we have an arranged data set, a data set that is sorted according to the threshold variable. After that we choose the value of the threshold variable that corresponds to the 15th percentile of the sample. Taking this value as the threshold value we split the whole sample into two sub samples and maximize the corresponding log likelihood function. We repeat this procedure up to the 85th percentile of the sample. This means that we have an adequate number of observations in the two sub samples in each case. The threshold value and the estimation of the parameter vector $\hat{\theta}$ will be these values that correspond to the maximum value of the maximized log likelihood functions for each possible value of the threshold parameter.

To test model linearity ($H_0: \beta = \beta'; \gamma = \gamma'; \delta = \delta'$ vs $H_1: \beta \neq \beta'; \gamma \neq \gamma'; \delta \neq \delta'$), we estimate the likelihood ratio (LR) test statistic:

$$LR = -2(l_{restricted} - l_{unrestricted})$$

where $l_{restricted}$ and $l_{unrestricted}$ are the maximum likelihood functions for the restricted and unrestricted model respectively. Since the threshold parameter is not identified under the null hypothesis, this statistic has a non conventional distribution tending towards the supremum of a chi-square process. This asymptotic distribution can be replicated by bootstrap according to Hansen (1996). Following this approach we perform a non-parametric bootstrap simulation procedure in order to derive the empirical significance levels of this test statistic. The non parametric bootstrap that we apply has the following steps: 1. Estimate the restricted model using the full sample of T observations and store the residuals and the fitted values of the interest rate; 2. draw with equal probability and with replacement (bootstrap) from the vector of the residuals to make another $T \times 1$ vector of residuals; 3. add this vector to the vector of fitted values of the interest rate obtained in step 1 to obtain an artificial vector of interest rate observations; 4. estimate the restricted and the unrestricted models using the artificial interest rate vector and construct a value of the LR statistic; 5. repeat steps 2-4 one thousand times. This gives one thousand simulated values for the LR statistic. The percentage of occasions that the simulated values of the LR statistic exceed the actual value of the LR statistic then corresponds to the empirical marginal significance level of the actual statistic.

We also construct the 95% confidence intervals for the threshold parameters. These confidence intervals are derived through the above bootstrap procedure. The results are summarized in Table 4.

The estimations show an asymmetric response of the central bank depending on the actual state of the economy. The likelihood ratio tests are in favor of rejecting the null hypothesis of linearity whether the output gap or the unemployment rate is used as a threshold variable. Regarding the output gap as the threshold variable we see that the first regime corresponds to output gap values less than or equal to 0.718%. We notice that the parameter of inflation deviation is statistically significant and below unity while the parameters of the output gap and the percentage change of the real exchange rate are not statistically significant. In the second regime which corresponds to output gap values greater than 0.718% all the parameters are statistically significant. More specifically the monetary authorities respond strongly to inflation deviations (as the corresponding coefficient is 2.314>1) but weakly to the output gap (0.828) and

* The derivation of the log likelihood function is demonstrated in the Appendix.

the real exchange rate (0.495). The smoothing parameter remains high (0.895). We also see that the 95% confidence interval for the threshold parameter, $[-2.536, 2.22]$, includes zero.

Considering the unemployment rate as a threshold variable the central bank's asymmetric response pertaining to the inflation deviation, the output gap and the percentage change in the real exchange rate still exists. The estimated threshold value is 3.182%. In the first regime that corresponds to unemployment rate values of the previous quarter less than or equal to the threshold value, the parameters of the inflation, the output gap and the exchange rate are statistically significant. The reaction of the Bank of Japan is more strongly to output gap than inflation deviations (2.69 versus 1.34) while it is weak to the real exchange rate (0.436). In the second regime all the parameters of interest are not statistically significant a fact that means that the monetary authorities abandon the monetary rule. The smoothing parameter is higher than that of the previous cases (0.963). The 95% confidence interval of the threshold parameter, $[2.005, 4.648]$, includes the mean value of the unemployment rate.

Figure 1 entails the graphs of the output gap and the unemployment rate including their corresponding threshold values. We notice that the break quarters that are estimated by the Bai-Perron tests are captured by this nonlinear model. More specifically, when we regard the output gap as the threshold variable we see that our model captures the breaks in 1980:3 and 1981:3 and is very close to the break of 1991:2. On the other hand, when we allow the unemployment rate to be the threshold variable our nonlinear model seems to not follow the previous path. In this case the only break is captured in 1995:3 and is not according to that estimated by the Bai-Perron tests. As we see, the patterns do not indicate a strong relationship between the output gap and the unemployment rate. Indeed, the correlation coefficient is relatively low ($\rho_{\hat{y},u} = -0.465$) indicating that the responses of the monetary authorities do not weighted equally to each threshold variable.

4. Conclusions

The famous Taylor rule is a good instrument rule for the case of a closed economy. In an open economy this rule has to be modified in order to include the external constraints of the home country. For this reason, the percentage change of the real exchange rate is added to the classical monetary rule. The estimating adjusted monetary rule is subject to significant nonlinearity, a nonlinearity that is associated with the real side of the economy. More specifically, when the output gap lies below or the unemployment rate exceeds its threshold value the monetary rule don't be followed by the authorities. When the opposite happens, the central banks' reactions obey the monetary rule.

There are many extensions of this study that can be followed. A forward looking version of the proposed model may be estimated allowing for endogenous variables and an exogenous threshold variable. One can examine threshold type models taking as a threshold variable the inflation deviation from its target or the volatility of inflation or output gap. Also you may allow as a threshold an endogenous variable (the expected inflation or the expected output gap) applying the method of threshold regression with endogenous threshold variable proposed by Kourtellis, Stengos and Tan (2007). Extension of our study in central banks' reaction functions that have an official inflation target zone such as the ECB may be very interesting.

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Appendix

A. The derivation of the Likelihood Function for the switching regime models

In this section we show the way that the likelihood function be set. We consider a process known as i.i.d. mixture distribution which is described in Hamilton (1994). Let's consider the unobserved random variable s_t which corresponds to the regime that a given process is in at date t . We assume that the variable s_t can take two values ($s_t = 1, 2$) and that $\varepsilon_{1t} \sim N(0, \sigma^2)$ and $\varepsilon_{2t} \sim N(0, \sigma'^2)$. The multivariate normal density for ε_j

is $f(\varepsilon_j) = \frac{1}{(2\pi\sigma_j^2)^{T_j/2}} \exp\left[-\frac{1}{2} \frac{\varepsilon_j' \varepsilon_j}{\sigma_j^2}\right]$ and the multivariate density for y_j conditional on the matrix of

regressors X is $f(y_j / X) = f(\varepsilon_j) \cdot \left| \frac{\partial \varepsilon_j}{\partial y_j} \right|$, where $\left| \frac{\partial \varepsilon_j}{\partial y_j} \right| = I_{T_j}$, for $j = 1, 2$. When the process is in regime 1

we assume that the observed variable $y_t \sim N(\mu_1, \sigma_1^2)$ and when it is in regime 2 then the observed variable $y_t \sim N(\mu_2, \sigma_2^2)$. So, the density of y_t conditional on the random variable s_t is

$$f(y_t / s_t = j; \theta) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_j)^2}{\sigma_j^2}\right], \text{ for } j = 1, 2.$$

The vector θ is a vector of population parameters. More specific we have $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \pi_1, \pi_2)'$. We denote the unconditional probability that s_t takes on the value j as $P(s_t = j; \theta) = \pi_j$, for $j = 1, 2$. The joint density-distribution function of y_t and s_t is given by the expression

$$\begin{aligned} P(y_t, s_t = j; \theta) &= f(y_t / s_t = j; \theta) \cdot P(s_t = j; \theta) = \\ &= \frac{\pi_j}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_j)^2}{\sigma_j^2}\right] \end{aligned}$$

The unconditional density of y_t is derived by summing the previous expression over all possible values for j :

$$f(y_t; \theta) = \sum_{j=1}^2 P(y_t, s_t = j; \theta) =$$

$$= \frac{\pi_1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_1)^2}{\sigma_1^2}\right] + \frac{\pi_2}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_2)^2}{\sigma_2^2}\right]$$

If the regime variable s_t is distributed as i.i.d. across different dates t , then the log likelihood for the observed data is given by the following expression:

$$L(\theta) = \sum_{t=1}^T \log f(y_t; \theta) =$$

$$= \sum_{t=1}^T \log \left\{ \frac{\pi_1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_1)^2}{\sigma_1^2}\right] + \frac{\pi_2}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{1}{2} \frac{(y_t - \mu_2)^2}{\sigma_2^2}\right] \right\}$$

If we are certain which observations come from regime j and which don't then the $P(s_t = j; \theta) = 1$ for those observations that come from regime j . So, the probability will be equal to zero for the observations that come from the other regime. Under this reasoning we'll have T_1 observations in the first regime with probability $\pi_1 = 1$ and $T_2 = T - T_1$ observations in the second regime with probability $\pi_2 = 1$. The log likelihood then takes the following form:

$$L(\theta) = \sum_{g=1}^{T_1} \log \left\{ \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{1}{2} \frac{(y_g - \mu_1)^2}{\sigma_1^2}\right] + 0 \right\}$$

$$+ \sum_{l=1}^{T_2} \log \left\{ 0 + \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{1}{2} \frac{(y_l - \mu_2)^2}{\sigma_2^2}\right] \right\}$$

or

$$L(\theta) = \left(-\frac{T_1}{2} \log(2\pi\sigma_1^2) - \frac{1}{2\sigma_1^2} \varepsilon_1' \varepsilon_1 \right) + \left(-\frac{T_2}{2} \log(2\pi\sigma_2^2) - \frac{1}{2\sigma_2^2} \varepsilon_2' \varepsilon_2 \right)$$

where $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)'$.

B. Tables

Table 1 Descriptive Statistics

Variables	U.S.A.		JAPAN	
	Mean	Std. Dev.	Mean	Std. Dev.
Nominal Interest Rate	6.668	3.6	4.216	3.558
Real Interest Rate	2.717	2.856	1.665	2.91
Inflation	4.094	2.522	2.551	4.61
Output gap	-0.608	2.064	-0.155	2.27
Unemployment Rate	6.209	1.407	2.952	1.166

Table 2. Bank of Japan reaction function. Baseline Estimates

$$i_t = (1 - \rho) \left[a + \beta (\pi_{t-1} - \pi^*) + \gamma \tilde{y}_{t-1} + \delta z_{t-1} \right] + \rho i_{t-1} + \varepsilon_t$$

Parameters	Estimation
a	4.216

<i>Parameters</i>	<i>Estimation</i>
ρ	0.9*** (21.24)
β	0.78*** (3.988)
γ	0.754* (1.714)
δ	0.141 (1.13)
π^*	2.698*** (2.456)
R^2	0.967
M.S.E.	0.416
Theil	0.117

t-statistics are included in parenthesis. Newey-West covariance matrix is used for standard errors. ***, **, * denotes 1%, 5%, 10% significance.

Table 3. Bai-Perron Tests for Bank of Japan's reaction function

<i>MODEL</i>	<i>BIC</i>	<i>Sequential</i>
Partial Structural	1980:3 [80:1,81:2]	1981:3 [81:1,86:2]
	1991:2 [91:1,92:4]	1991:2 [90:4,93:1]
Pure Structural	1981:3 [81:2,83:4]	1981:3 [81:2,82:4]
		1991:2 [91:1,92:1]

Table 4. Bank of Japan. Estimated Reaction Functions with unknown threshold value

$$i_t = \begin{cases} (1 - \rho) [a + \beta (\pi_{t-1} - \pi^*) + \gamma \tilde{y}_{t-1} + \delta z_{t-1}] + \rho i_{t-1} + \varepsilon_{1t}, & \text{if } q_{t-1} \leq \bar{q} \\ (1 - \rho) [a + \beta' (\pi_{t-1} - \pi^*) + \gamma' \tilde{y}_{t-1} + \delta' z_{t-1}] + \rho i_{t-1} + \varepsilon_{2t}, & \text{if } q_{t-1} > \bar{q} \end{cases}$$

<i>Parameters</i>	Threshold variable: Output gap ($\tilde{y}_{t-1}$)	Threshold variable: Unemployment rate (u_{t-1})
α	2.948*** (4.71)	0.461 (0.35)
β	0.579*** (6.8)	1.34*** (2.92)
γ	0.034 (0.104)	2.69*** (2.828)

Parameters	Threshold variable: Output gap ($\tilde{y}_{t-1}$)	Threshold variable: Unemployment rate (u_{t-1})
δ	0.055 (1.53)	0.436*** (2.426)
β'	2.314*** (4.423)	-0.003 (0.008)
γ'	0.828* (1.751)	0.233 (1.12)
δ'	0.495*** (3.683)	-0.026 (0.865)
ρ	0.895*** (47.15)	0.963*** (116.84)
σ^2	0.171*** (6.94)	0.618*** (6.4)
σ'^2	0.63*** (4.56)	0.004*** (4.15)
LR test (p-value)	68.549 (0.002)	161.89 (0.00)
Threshold value	$\bar{q}_{\tilde{y}_{t-1}} = 0.7175$ [-2.536, 2.22]	$\bar{q}_{u_{t-1}} = 3.182$ [2.005, 4.648]
Max likelihood	$\ell = -101.948$	$\ell = -55.276$
Percentile	0.698	0.684
Theil	0.101	0.118
M.S.E.	0.31	0.424

t-statistics are included in parenthesis. ***, **, * denotes 1%, 5%, 10% significance.

Figure 1

