

ESTIMATION OF $P(X>Y)$ IN CASE OF BIVARIATE NORMAL DISTRIBUTION USING RANKED SET SAMPLING WITH CONCOMITANT VARIABLE

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Abstract

Nonparametric-type estimation of $\theta = P(X > Y)$, based on ranked set sampling technique with concomitant random variable, is considered. The maximum likelihood estimation of θ is also considered using RSS (ranked set sampling) and MRSS (median ranked set sampling) techniques with concomitant random variable. The estimators obtained are compared to their counterparts based on simple random sampling (SRS) using bias and MSE (mean square error). It appears that the suggested nonparametric estimator based on RSS is more efficient than that based on SRS and the MLE estimator based on MRSS is more efficient than the MLE based on RSS which is in turn more efficient than the MLE based on SRS.

Key Words: Simple Random Sampling, Ranked Set Sampling, Median Ranked Set Sampling, Concomitant Variable, Nonparametric Estimation; Maximum Likelihood Estimation, Bias, Mean Square Error.

1. Introduction

The ranked set sampling (RSS) technique was first suggested by McIntyre in a paper in 1952. It consists of choosing randomly m sets of size m each from the population of interest; the elements in each set are ranked by judgment with respect to the variable of interest. Then, the i^{th} judgment ranked unit from the i^{th} set is chosen for actual quantifications. In practice, the set size m should be small to avoid ranking errors; a larger sample size can be obtained by iterating the procedure. The RSS procedure and its variations can be applied in many areas such as forestry, agriculture, animal science, medicine, etc. For details about these applications, see Halls and Dell (1966), Martin et al. (1980), Al-Saleh and Al-Shrafat (2001). Takahasi and Wakimoto (1968) established a rigorous statistical foundation of the theory of ranked set sampling. Other related work on RSS can be found in Patil et al. (1999); Kaur et al. (1995); Al-Saleh and Zheng (2002) and Barabesi and Pisani (2004).

Stokes (1980) introduced a modified ranked set sampling procedure in which only the largest or the smallest judgment ranked unit is chosen for quantification. The method was investigated further by Samawi et al. (1996).

In this paper, first we propose to use the RSS with concomitant variable for nonparametric estimation of $\theta = P(X > Y)$. Then the MLEs of θ and modifications of them based on RSS and median ranked set sampling (MRSS) with a concomitant variable are considered. Their properties are studied. The efficiency of the estimators is compared with those obtained from SRS.

2. RSS and MRSS with concomitant variable

Assume that (X, Y) is a bivariate random vector such that variable Y is difficult to measure or to order by judgment, but the concomitant variable X , which is correlated with Y , is easier to measure or to order by judgment. The variable X may be used to acquire the rank of Y as follows:

- Select m units from the jpdf of X and Y using m SRS's of size m each. Identify by judgment the i^{th} ranked observation of the i^{th} sample to the variable X .
 - Repeat the above two steps r times, if necessary, until the desired sample size, $n = r m$ is obtained.
 - Measure accurately the selected n judgment identified units for both variables.
- The set of the n pairs obtained using the above procedure, is called a RSS with concomitant variable.

If m is odd, the MRSS can be obtained as in the steps above, but in step (a) we measure always the $(\frac{m+1}{2})^{th}$ ranked observation and if m is even then $\frac{m}{2}$ times we measure the $(\frac{m}{2})^{th}$ ranked observation and $\frac{m}{2}$ times we measure the $(\frac{m}{2} + 1)^{th}$ ranked observation. Repeat this process r times until the desired sample size is obtained.

Let $\{(X_{(i:m)j}, Y_{[i:m]j}), i = 1, 2, \dots, m, j = 1, 2, \dots, r\}$ be a RSS from $f_{X,Y}(x, y)$ based on the concomitant variable X . If judgment ranking is perfect then, $X_{(i:m)j}$ and $Y_{[i:m]j}$ are, respectively, the i^{th} smallest value of X from the i^{th} sample of the j^{th} cycle and the corresponding value of Y , where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, r$. Note that the pairs of this sample are independent but not identically distributed. The jpdf of $(X_{(i:m)j}, Y_{[i:m]j})$ is denoted by $f_{(i:m)j}(x, y)$ and given by:

$$f_{(i:m)j}(x, y) = f_{X_{(i:m)}}(x) f_{Y|X}(y|x)$$

where $f_{X_{(i:m)j}}$ is the pdf of the i^{th} order statistic of a SRS of size m from the marginal distribution of X and $f_{Y|X}(y|x)$ is the conditional density of Y given X , (see Yang, 1977 and Stokes, 1980).

3. Nonparametric Estimation of $\theta = P(X > Y)$

We will consider the estimation of $\theta = P(X > Y)$ nonparametrically. The nonparametric estimator of θ based on the SRS $(X_1, Y_1), \dots, (X_n, Y_n)$ is given by:

$$\hat{\theta}_{SRS} = \frac{1}{n} \sum_{i=1}^n I(X_i > Y_i)$$

Note that $\hat{\theta}_{SRS}$ is an unbiased estimator of θ with variance $\frac{\theta(1-\theta)}{n}$. We propose the following nonparametric estimator of θ based on the RSS with concomitant variable $\{(X_{(i:m)j}, Y_{[i:m]j}), i = 1, 2, \dots, m, j = 1, 2, \dots, r\}$:

$$\hat{\theta}_{RSS} = \frac{1}{rm} \sum_{j=1}^r \sum_{i=1}^m I(X_{(i:m)j} > Y_{[i:m]j})$$

where $n = r m$. Then we have the following result where the proofs of (a) and (c) will be given.

Theorem 1:

(a) $\hat{\theta}_{RSS}$ is an unbiased estimator of θ .

$$(b) Var(\hat{\theta}_{RSS}) = \frac{1}{n^2} \sum_{j=1}^r \sum_{i=1}^m \theta_{ij} (1 - \theta_{ij})$$

$$(c) e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS}) = \frac{var(\hat{\theta}_{SRS})}{var(\hat{\theta}_{RSS})} = \frac{m\theta(1-\theta)}{\sum_{i=1}^m \theta_{i1}(1-\theta_{i1})} \geq 1$$

where $\theta_{ij} = P(X_{(i:m)j} > Y_{[i:m]j})$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, r$.

Proof:

$$\begin{aligned} (a) E(\hat{\theta}_{RSS}) &= \frac{1}{n} \sum_{j=1}^r \sum_{i=1}^m P(X_{(i:m)j} > Y_{[i:m]j}) \\ &= \frac{1}{n} \sum_{j=1}^r \sum_{i=1}^m \int_{-\infty}^{\infty} \int_y^{\infty} f_{X_{(i:m)}}(x) f_{Y|X}(y|x) dx dy \\ &= \int_{-\infty}^{\infty} \int_y^{\infty} \left\{ \frac{\sum_{j=1}^r \sum_{i=1}^m f_{X_{(i:m)}}(x)}{n} \right\} f_{Y|X}(y|x) dx dy \\ &= \int_{-\infty}^{\infty} \int_y^{\infty} f(x) f_{Y|X}(y|x) dx dy \end{aligned}$$

(by Takahasi and Wakimoto (1968))

$$= \int_{-\infty}^{\infty} \int_y^{\infty} f(x, y) dx dy = \theta$$

(c) By Cauchy Schwarz inequality, we have

$$\sum_{j=1}^r \sum_{i=1}^m \theta_{ij} \leq \sqrt{\sum_{j=1}^r \sum_{i=1}^m 1} \sqrt{\sum_{j=1}^r \sum_{i=1}^m \theta_{ij}^2}$$

which implies that

$$n\theta \leq \sqrt{n} \sqrt{\sum_{j=1}^r \sum_{i=1}^m \theta_{ij}^2}$$

which in turn implies

$$\begin{aligned} Var(\hat{\theta}_{RSS}) &= \frac{1}{n^2} \sum_{j=1}^r \sum_{i=1}^m \theta_{ij} (1 - \theta_{ij}) = \frac{1}{n^2} \left[\sum_{j=1}^r \sum_{i=1}^m \theta_{ij} - \sum_{j=1}^r \sum_{i=1}^m \theta_{ij}^2 \right] \\ &= \frac{1}{n^2} [n\theta - n\theta^2] = \frac{\theta(1-\theta)}{n} \\ &= Var(\hat{\theta}_{SRS}) \end{aligned}$$

which ends the proof of (c).

Note that from Theorem (1), $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ does not depend on r . Therefore in this section, we will assume that $r=1$. Also, note that $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ depends on the joint distribution of X and Y . We will assume that X and Y have a bivariate normal distribution $BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ where σ_x^2, σ_y^2 and ρ are

known. We may assume without loss of generality that $\sigma_x^2 = \sigma_y^2 = 1$. Note that $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ depends on μ_x and μ_y through $\mu_x - \mu_y$. Table 1 gives $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ for $m=2, 3, 4, 5$, $\rho = -.8, -.5, -.3, .3, .5, .8$ and $\mu_x - \mu_y = -1.00, -0.75, -0.62, -0.3, 0.3, 0.47, 0.85, 1.57$. FORTRAN is used for the calculations in this paper.

The following can be deduced from Table 1 :

- (a) The efficiency $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ is always greater than one.
- (b) The efficiency $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ increases in m for negative ρ for fixed ρ and fixed $\mu_x - \mu_y$ but not for positive ρ .
- (c) The efficiency $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ in case ρ is negative is better than when ρ is positive.
- (d) For fixed m and fixed $\mu_x - \mu_y$, the efficiency $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ increases as $|\rho|$ increases for negative ρ and decreases as ρ increases for positive ρ .
- (e) For fixed ρ and fixed $\mu_x - \mu_y$, the efficiency $e(\hat{\theta}_{RSS}, \hat{\theta}_{SRS})$ increases as $|\mu_x - \mu_y|$ decreases for negative ρ .
- (f) Based on Table 1, we may conclude that $\hat{\theta}_{RSS}$ is a good competitor of $\hat{\theta}_{SRS}$.

4. MLE of $\theta = P(X > Y)$

We will consider the MLE $\theta = P(X > Y)$ using SRS and RSS and MRSS with concomitant random variable in case of the bivariate normal distribution $BVN(\mu_x, \mu_y, 1, 1, \rho)$ where ρ is known. Denote the density and the cumulative density of the standard normal distribution by ϕ and Φ respectively. Note that

$$\theta = P(X > Y) = \Phi\left[\frac{\mu_x - \mu_y}{\sqrt{2(1-\rho)}}\right]$$

which implies that the MLE of θ will be $\hat{\theta} = \Phi\left[\frac{\hat{\mu}_x - \hat{\mu}_y}{\sqrt{2(1-\rho)}}\right]$ where $\hat{\mu}_x$ and $\hat{\mu}_y$ are MLEs of μ_x and μ_y respectively.

It is easy to show that based on the SRS $(X_1, Y_1), \dots, (X_n, Y_n)$, the MLEs of μ_x and μ_y are $\bar{X}$ and $\bar{Y}$ respectively where $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ and $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$. This implies that the MLE of θ using the SRS is:

$$\hat{\theta}_{SRS} = \Phi\left[\frac{\bar{X} - \bar{Y}}{\sqrt{2(1-\rho)}}\right] \quad (4.1)$$

If $\{(X_{(i:m)j}, Y_{[i:m]j}), i = 1, 2, \dots, m, j = 1, 2, \dots, r\}$ is a RSS from $BVN(\mu_x, \mu_y, 1, 1, \rho)$ based on the concomitant variable X , then it can be shown as in Stokes, S. (1980) and Al-Saleh and Al-Ananbeh (2007) that the maximum likelihood estimators of μ_x and μ_y exist and are unique, say $\hat{\mu}_{x,RSS}$ and $\hat{\mu}_{y,RSS}$ and they are given as the solutions of the following two equations:

$$\mu_x - \bar{x}_{RSS} + \frac{1}{n} \sum_{j=1}^r \sum_{i=1}^m (i-1) \left\{ \frac{\varphi(x_{(i:m)j} - \mu_x)}{\Phi(x_{(i:m)j} - \mu_x)} - \frac{\varphi(x_{(i:m)j} - \mu_x)}{[1 - \Phi(x_{(i:m)j} - \mu_x)]} \right\} = 0 \quad (4.2)$$

$$\mu_y = \bar{y}_{RSS} - \rho (\bar{x}_{RSS} - \mu_x). \quad (4.3)$$

Hence an estimator of θ based on $\{(X_{(i:m)j}, Y_{[i:m]j}), i = 1, 2, \dots, m, j = 1, 2, \dots, r\}$ is given by

$$\hat{\theta}_{RSS} = \Phi\left[\frac{\hat{\mu}_{x,RSS} - \hat{\mu}_{y,RSS}}{\sqrt{2(1-\rho)}}\right] \quad (4.4)$$

Similarly, the maximum likelihood estimators of μ_x and μ_y based on MRSS with concomitant variable X can be shown to exist and are unique, say $\hat{\mu}_{x,MRSS}$ and $\hat{\mu}_{y,MRSS}$, and they are given by the solutions of the following two equations:

(a) If m is odd:

$$\mu_x - \bar{x}_{MRSS} + \left(\frac{m-1}{2}\right) \sum_{j=1}^r \sum_{i=1}^m \left\{ \frac{\varphi(x_{((\frac{m+1}{2})i:m)j} - \mu_x)}{\Phi(x_{((\frac{m+1}{2})i:m)j} - \mu_x)} - \frac{\varphi(x_{((\frac{m+1}{2})i:m)j} - \mu_x)}{[1 - \Phi(x_{((\frac{m+1}{2})i:m)j} - \mu_x)]} \right\} = 0 \quad (4.5)$$

$$\mu_y = \bar{y}_{MRSS} - \rho (\bar{x}_{MRSS} - \mu_x) \quad (4.6)$$

(b) If m is even:

$$\begin{aligned} \mu_x - \bar{x}_{MRSS} + \left(\frac{1}{n}\right) \sum_{j=1}^r \sum_{i=1}^{\left(\frac{m}{2}\right)} \left(\frac{m}{2} - 1\right) \left\{ \frac{\varphi(x_{((\frac{m}{2})i:m)j} - \mu_x)}{\Phi(x_{((\frac{m}{2})i:m)j} - \mu_x)} \right\} \\ + \left(\frac{1}{n}\right) \sum_{j=1}^r \sum_{i=((\frac{m}{2})+1)}^m \left(\frac{m}{2}\right) \left\{ \frac{\varphi(x_{((\frac{m}{2}+1)i:m)j} - \mu_x)}{\Phi(x_{((\frac{m}{2}+1)i:m)j} - \mu_x)} \right\} \\ - \left(\frac{1}{n}\right) \sum_{j=1}^r \sum_{i=1}^{\left(\frac{m}{2}\right)} \left(\frac{m}{2}\right) \left\{ \frac{\varphi(x_{((\frac{m}{2})i:m)j} - \mu_x)}{1 - \Phi(x_{((\frac{m}{2})i:m)j} - \mu_x)} \right\} \\ - \left(\frac{1}{n}\right) \sum_{j=1}^r \sum_{i=((\frac{m}{2})+1)}^m \left(\frac{m}{2} - 1\right) \left\{ \frac{\varphi(x_{((\frac{m}{2}+1)i:m)j} - \mu_x)}{1 - \Phi(x_{((\frac{m}{2}+1)i:m)j} - \mu_x)} \right\} = 0 \end{aligned} \quad (4.7)$$

$$\mu_y = \bar{y}_{MRSS} - \rho (\bar{x}_{MRSS} - \mu_x). \quad (4.8)$$

These imply that the estimator of θ based on MRSS with concomitant variable is given by

$$\hat{\theta}_{MRSS} = \Phi\left[\frac{\hat{\mu}_{x,MRSS} - \hat{\mu}_{y,MRSS}}{\sqrt{2(1-\rho)}}\right] \quad (4.9)$$

The MLEs under RSS and MRSS with concomitant variable can't be obtained in closed form. In order to get a closed form of approximate MLEs of μ_x and μ_y , we replace the last terms in the left hand sides of

equations (4.2), (4.5) and (4.7) by their expectations. This technique was used by Mehrotra and Nanda (1974) for censored data, Zheng and Al-Saleh (2002) for MLE using RSS, and Abu-Dayyeh and Al-Sawi (2007) using moving extreme ranked set sampling. But, it can be shown easily that the expectation of each of these terms is zero. Therefore, the MMLEs (modified MLEs) for μ_x and μ_y are respectively given by the means of the x's and y's under the sampling scheme under consideration. Hence, the MMLE of θ is given as follows:

(a) Under RSS with concomitant variable it is given by:

$$\hat{\theta}_{Mod,RSS} = \Phi\left[\frac{\bar{x}_{RSS} - \bar{y}_{RSS}}{\sqrt{2(1-\rho)}}\right] \quad (4.10)$$

(b) Under MRSS with concomitant variable it is given by:

$$\hat{\theta}_{Mod,MRSS} = \Phi\left[\frac{\bar{x}_{MRSS} - \bar{y}_{MRSS}}{\sqrt{2(1-\rho)}}\right] \quad (4.11)$$

5. Simulation

We compared the estimators $\hat{\theta}_{SRS}$, $\hat{\theta}_{RSS}$, $\hat{\theta}_{MRSS}$, $\hat{\theta}_{Mod,RSS}$ and $\hat{\theta}_{Mod,MRSS}$ by using their biases and MSEs. The biases and MSEs depend on the number of cycles (r), the set size (m), μ_x , μ_y (only through $\mu_x - \mu_y$) and ρ . An extensive simulation was conducted using a program written using FORTRAN and routines from NAG libraries. Several combinations of the four parameters r, m, ρ , $\mu_x - \mu_y$ were considered. The parameter values considered were m=1, 2, 3, 4, 5 and 6, r=10, 15 and 20, ρ from -1.0 to 1.0 and μ_x and μ_y ranging from -4.0 to 4.0. The biases, estimates and efficiencies of the estimators $\hat{\theta}_{RSS}$, $\hat{\theta}_{MRSS}$, $\hat{\theta}_{Mod,RSS}$ and $\hat{\theta}_{Mod,MRSS}$ with respect to $\hat{\theta}_{SRS}$ are reported in Tables 2, 3, 4 for when r=20 and m=2, 3, 4 and $\rho = -.8, -.5, -.3, .3, .5, .8$ and $\mu_x - \mu_y = -1.00, -0.75, -0.62, -0.3, 0.3, 0.47, 0.85, 1.57$. The efficiency of the estimator $\hat{\theta}$ w.r.t the estimator $\hat{\theta}_{SRS}$ is defined by:

$$e(\hat{\theta}, \hat{\theta}_{SRS}) = \frac{MSE(\hat{\theta}_{SRS})}{var(\hat{\theta})}$$

The following can be deduced from Table 1 :

- (a) The biases of $\hat{\theta}_{MRSS}$ and $\hat{\theta}_{Mod,MRSS}$ are equal and the biases of $\hat{\theta}_{RSS}$ and $\hat{\theta}_{Mod,RSS}$ are equal.
- (b) The bias of $\hat{\theta}_{MRSS}$ is smaller than that of $\hat{\theta}_{RSS}$ which is in turn smaller than that of $\hat{\theta}_{SRS}$.
- (c) The biases of all estimators are very small. So, we may consider all of them as unbiased estimators of θ .
- (d) The efficiencies of all the estimators are greater than one.
- (e) The efficiencies of all estimators increase in m for fixed ρ , r and $\mu_x - \mu_y$.
- (f) The efficiencies of all estimators is in case ρ is negative are better than when ρ is positive.
- (g) Based on Tables 2-7, we may conclude that $\hat{\theta}_{RSS}$, $\hat{\theta}_{MRSS}$, $\hat{\theta}_{Mod,RSS}$ and $\hat{\theta}_{Mod,MRSS}$ are real competitors of $\hat{\theta}_{SRS}$.

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Tables

Table 1: Efficiency of $\hat{\theta}_{RSS}$ w.r.t. $\hat{\theta}_{SRS}$

$\mu_x - \mu_y$	m=2	m=3	m=4	m=5	m=2	m=3	m=4	m=5
	$\rho = -0.8$				$\rho = 0.8$			
-1.00	1.1047	1.1931	1.2477	1.2956	1.0154	1.0022	1.0141	1.0152
-0.75	1.1097	1.1962	1.2598	1.3154	1.0055	1.0044	1.0054	1.0094
-0.62	1.1262	1.2087	1.2812	1.3190	1.0132	1.0169	1.0039	1.0076
-0.30	1.1317	1.2198	1.2791	1.3412	1.0122	1.0154	1.0097	1.0177
0.30	1.1304	1.2187	1.2771	1.3281	1.0124	1.0182	1.0154	1.0227
0.47	1.1294	1.2090	1.2756	1.3251	1.0064	1.0137	1.0124	1.0173
0.85	1.1206	1.2005	1.2595	1.3010	1.0023	1.0018	1.0009	1.0087
1.57	1.0836	1.1474	1.2030	1.2344	1.0037	1.0371	1.0011	1.0174
	$\rho = -0.5$				$\rho = 0.5$			
-1.00	1.0841	1.1459	1.1782	1.2093	1.0204	1.0325	1.0263	1.0391
-0.75	1.0826	1.1470	1.1919	1.2188	1.0119	1.0351	1.0409	1.0415
-0.62	1.0953	1.1533	1.1984	1.2318	1.0213	1.0419	1.0470	1.0475
-0.30	1.1022	1.1645	1.2068	1.2418	1.0242	1.0372	1.0558	1.0518
0.30	1.1016	1.1632	1.2132	1.2305	1.0268	1.0420	1.0524	1.0623
0.47	1.0887	1.1642	1.1931	1.2394	1.0291	1.0311	1.0488	1.0606
0.85	1.0876	1.1428	1.1840	1.2341	1.0207	1.0309	1.0486	1.0488
1.57	1.0678	1.1068	1.1442	1.1734	1.0130	1.0266	1.0177	1.0243

	$\rho = -0.3$				$\rho = 0.3$			
-1.00	1.0623	1.1122	1.1390	1.1687	1.0350	1.0467	1.0658	1.0677
-0.75	1.0777	1.1187	1.1533	1.1798	1.0313	1.0515	1.0710	1.0737
-0.62	1.0721	1.1234	1.1563	1.1831	1.0418	1.0582	1.0691	1.0875
-0.30	1.0817	1.1339	1.1575	1.1901	1.0397	1.0631	1.0764	1.0844
0.30	1.0844	1.1309	1.1704	1.1866	1.0274	1.0589	1.0789	1.0727
0.47	1.0763	1.1290	1.1643	1.1844	1.0388	1.0519	1.0753	1.0832
0.85	1.0794	1.1072	1.1499	1.1735	1.0316	1.0513	1.0666	1.0639
1.57	1.0450	1.0796	1.1137	1.1205	1.0121	1.0266	1.0451	1.0227

Table 2: Biases and Efficiencies of the Estimators, m=2, r=20

ρ	$\mu_x - \mu_y$	B-SRS	B-RSS	B-RSSM	B-MRSS	B-MRSSM	SRS /RSS	SRS /RSSM	SRS /MRSS	SRS/ MRSSM
-0.8	-1.00	0.0021	0.0017	0.0017	0.0017	0.0017	1.4088	1.3996	1.4088	1.3996
	-0.75	0.0021	0.0013	0.0013	0.0013	0.0013	1.4107	1.4006	1.4107	1.4006
	-0.62	0.0014	0.0012	0.0012	0.0012	0.0012	1.4098	1.3990	1.4098	1.3990
	-0.30	0.0007	0.0003	0.0003	0.0003	0.0003	1.4067	1.3958	1.4067	1.3958
	0.30	-0.0009	-0.0003	-0.0003	-0.0033	-0.0003	1.4149	1.4033	1.4149	1.4033
	0.47	-0.0017	-0.0009	-0.0009	-0.0009	-0.0009	1.4050	1.3928	1.4050	1.3928
	0.85	-0.0020	-0.0015	-0.0015	-0.0015	-0.0015	1.4057	1.3963	1.4057	1.3963
	1.51	-0.0030	-0.0021	-0.0021	-0.0021	-0.0021	1.4347	1.4266	1.4347	1.4266
-0.3	-1.00	0.0023	0.0019	0.0019	0.0019	0.0019	1.2693	1.2645	1.2693	1.2645
	-0.75	0.0018	0.0017	0.0017	0.0017	0.0017	1.2653	1.2592	1.2653	1.2592
	-0.62	0.0015	0.0013	0.0013	0.0013	0.0013	1.2700	1.2653	1.2700	1.2653
	-0.30	0.0013	0.0008	0.0008	0.0008	0.0008	1.2555	1.2482	1.2555	1.2482
	0.30	-0.0012	-0.0011	-0.0011	-0.0042	-0.0011	1.2524	1.2465	1.2524	1.2465
	0.47	-0.0008	-0.0011	-0.0011	-0.0011	-0.0011	1.2578	1.2519	1.2578	1.2519
	0.85	-0.0023	-0.0015	-0.0015	-0.0015	-0.0015	1.2666	1.2606	1.2666	1.2606
	1.51	-0.0031	-0.0024	-0.0024	-0.0024	-0.0024	1.2854	1.2798	1.2854	1.2798
0.3	-1.00	0.0028	0.0026	0.0025	0.0026	0.0025	1.1064	1.1041	1.1064	1.1041
	-0.75	0.0026	0.0020	0.0020	0.0020	0.0020	1.1320	1.1290	1.1320	1.1290
	-0.62	0.0024	0.0023	0.0023	0.0023	0.0023	1.1104	1.1073	1.1104	1.1073
	-0.30	0.0008	0.0014	0.0014	0.0014	0.0014	1.1179	1.1156	1.1179	1.1156
	0.30	-0.0015	-0.0013	-0.0013	-0.0048	-0.0013	1.1292	1.1258	1.1292	1.1258
	0.47	-0.0016	-0.0020	-0.0020	-0.0020	-0.0020	1.1076	1.1051	1.1076	1.1051
	0.85	-0.0030	-0.0027	-0.0026	-0.0027	-0.0026	1.1296	1.1274	1.1296	1.1274
	1.51	-0.0026	-0.0025	-0.0025	-0.0025	-0.0025	1.1238	1.1201	1.1238	1.1201
0.8	-1.00	0.0023	0.0020	0.0020	0.0020	0.0020	1.0440	1.0435	1.0440	1.0435
	-0.75	0.0033	0.0026	0.0026	0.0026	0.0026	1.0522	1.0515	1.0522	1.0515
	-0.62	0.0029	0.0030	0.0030	0.0030	0.0030	1.0265	1.0258	1.0265	1.0258
	-0.30	0.0024	0.0024	0.0024	0.0024	0.0024	1.0301	1.0291	1.0301	1.0291
	0.30	-0.0024	-0.0017	-0.0018	-0.0051	-0.0018	1.0303	1.0292	1.0303	1.0292
	0.47	-0.0028	-0.0026	-0.0026	-0.0026	-0.0026	1.0291	1.0287	1.0291	1.0287
	0.85	-0.0029	-0.0025	-0.0025	-0.0025	-0.0025	1.0562	1.0557	1.0562	1.0557
	1.51	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	1.0251	1.0237	1.0251	1.0237

Table 3: Biases and Efficiencies of the Estimators, m=3, r=20

ρ	$\mu_x - \mu_y$	B-SRS	B-RSS	B-RSSM	B-MRSS	B-MRSSM	SRS /RSS	SRS /RSSM	SRS /MRSS	SRS/ MRSSM
-0.8	-1.00	0.0014	0.0010	0.0010	0.0009	0.0009	1.7723	1.7365	1.9824	1.9817
	-0.75	0.0015	0.0007	0.0007	0.0006	0.0006	1.7556	1.7249	1.9742	1.9737
	-0.62	0.0009	0.0005	0.0006	0.0005	0.0005	1.7848	1.7503	1.9750	1.9744
	-0.30	0.0007	0.0003	0.0002	0.0001	0.0001	1.7865	1.7523	1.9749	1.9745
	0.30	-0.0007	-0.0001	-0.0002	0.0000	0.0000	1.7704	1.7368	1.9667	1.9663
	0.47	-0.0005	-0.0002	-0.0002	-0.0003	-0.0003	1.8001	1.7646	1.9809	1.9803
	0.85	-0.0010	-0.0009	-0.0009	-0.0007	-0.0007	1.7489	1.7127	1.9595	1.9585
	1.51	-0.0020	-0.0011	-0.0011	-0.0010	-0.0010	1.7490	1.7159	1.9573	1.9567
-0.3	-1.00	0.0017	0.0011	0.0011	0.0009	0.0009	1.4700	1.4535	1.5554	1.5552
	-0.75	0.0014	0.0011	0.0011	0.0012	0.0012	1.4853	1.4682	1.5647	1.5644
	-0.62	0.0012	0.0007	0.0006	0.0008	0.0008	1.4652	1.4486	1.5543	1.5543

	-0.30	0.0010	0.0004	0.0005	0.0008	0.0008	1.4624	1.4432	1.5692	1.5691
	0.30	-0.0004	-0.0001	-0.0002	-0.0001	-0.0001	1.4459	1.4278	1.5423	1.5423
	0.47	-0.0004	-0.0006	-0.0006	-0.0005	-0.0005	1.4480	1.4303	1.5510	1.5506
	0.85	-0.0018	-0.0010	-0.0010	-0.0010	-0.0010	1.4447	1.4272	1.5662	1.5660
	1.51	-0.0022	-0.0017	-0.0017	-0.0015	-0.0015	1.4731	1.4569	1.5742	1.5739
0.3	-1.00	0.0022	0.0015	0.0015	0.0018	0.0018	1.2162	1.2084	1.2387	1.2385
	-0.75	0.0016	0.0013	0.0013	0.0016	0.0016	1.1941	1.1888	1.2189	1.2188
	-0.62	0.0015	0.0014	0.0014	0.0010	0.0010	1.2210	1.2134	1.2544	1.2544
	-0.30	0.0009	0.0009	0.0009	0.0006	0.0006	1.2006	1.1942	1.2378	1.2378
	0.30	-0.0010	-0.0009	-0.0009	-0.0005	-0.0005	1.2138	1.2050	1.2532	1.2532
	0.47	-0.0010	-0.0008	-0.0008	-0.0009	-0.0009	1.1995	1.1925	1.2412	1.2410
	0.85	-0.0020	-0.0020	-0.0020	-0.0017	-0.0017	1.2258	1.2201	1.2629	1.2629
	1.51	-0.0018	-0.0015	-0.0015	-0.0015	-0.0015	1.2043	1.1977	1.2629	1.2629
0.8	-1.00	0.0015	0.0014	0.0014	0.0014	0.0014	1.0398	1.0384	1.0574	1.0574
	-0.75	0.0019	0.0019	0.0018	0.0018	0.0018	1.0479	1.0463	1.0624	1.0624
	-0.62	0.0021	0.0016	0.0016	0.0017	0.0017	1.0611	1.0593	1.0675	1.0676
	-0.30	0.0014	0.0013	0.0014	0.0012	0.0012	1.0573	1.0552	1.0761	1.0761
	0.30	-0.0012	-0.0015	-0.0015	-0.0012	-0.0012	1.0503	1.0494	1.0560	1.0560
	0.47	-0.0019	-0.0014	-0.0014	-0.0016	-0.0016	1.0593	1.0582	1.0633	1.0634
	0.85	-0.0019	-0.0018	-0.0018	-0.0017	-0.0017	1.0550	1.0538	1.0651	1.0651
	1.51	-0.0004	-0.0003	-0.0003	-0.0004	-0.0004	1.0627	1.0611	1.0725	1.0725

Table 4: Biases and Efficiencies of the Estimators, m=4, r=20

ρ	$\mu_x - \mu_y$	B-SRS	B-RSS	B-RSSM	B-MRSS	B-MRSSM	SRS /RSS	SRS /RSSM	SRS /MRSS	SRS /MRSSM
-0.8	-1.00	0.0015	0.0007	0.0007	0.0007	0.0007	2.1386	2.0828	2.3585	2.3549
	-0.75	0.0007	0.0004	0.0004	0.0004	0.0004	2.0863	2.0256	2.3433	2.3396
	-0.62	0.0008	0.0003	0.0004	0.0002	0.0002	2.1159	2.0545	2.3585	2.3542
	-0.30	0.0004	0.0001	0.0001	0.0002	0.0002	2.1361	2.0742	2.3430	2.3375
	0.30	-0.0007	-0.0002	-0.0002	-0.0003	-0.0003	2.1612	2.1033	2.3923	2.3892
	0.47	-0.0007	-0.0001	-0.0001	0.0000	0.0000	2.0920	2.0297	2.3146	2.3101
	0.85	-0.0010	-0.0006	-0.0006	-0.0005	-0.0005	2.1078	2.0460	2.3434	2.3398
	1.51	-0.0014	-0.0007	-0.0007	-0.0006	-0.0006	2.1081	2.0454	2.3581	2.3531
-0.3	-1.00	0.0012	0.0008	0.0008	0.0006	0.0007	1.6347	1.6110	1.7317	1.7293
	-0.75	0.0007	0.0007	0.0007	0.0006	0.0006	1.6210	1.5940	1.7099	1.7078
	-0.62	0.0008	0.0006	0.0006	0.0005	0.0005	1.6124	1.5867	1.7027	1.7007
	-0.30	0.0005	0.0005	0.0005	0.0002	0.0002	1.6107	1.5840	1.7151	1.7132
	0.30	-0.0005	-0.0002	-0.0002	-0.0002	-0.0002	1.6050	1.5783	1.7205	1.7181
	0.47	-0.0012	0.0000	0.0000	-0.0001	-0.0001	1.6338	1.6061	1.7250	1.7234
	0.85	-0.0014	-0.0007	-0.0006	-0.0008	-0.0008	1.5995	1.5724	1.7041	1.7024
	1.51	-0.0014	-0.0010	-0.0010	-0.0010	-0.0010	1.6051	1.5785	1.7218	1.7209
0.3	-1.00	0.0011	0.0012	0.0012	0.0010	0.0010	1.2455	1.2382	1.2864	1.2864
	-0.75	0.0015	0.0008	0.0008	0.0009	0.0008	1.2571	1.2478	1.2950	1.2946
	-0.62	0.0008	0.0009	0.0009	0.0012	0.0012	1.2514	1.2428	1.2894	1.2887
	-0.30	0.0005	0.0006	0.0006	0.0005	0.0005	1.2377	1.2301	1.2757	1.2751
	0.30	-0.0007	-0.0005	-0.0005	-0.0004	-0.0004	1.2654	1.2555	1.2852	1.2848
	0.47	-0.0008	-0.0009	-0.0009	-0.0007	-0.0007	1.2559	1.2446	1.2995	1.2990
	0.85	-0.0015	-0.0012	-0.0012	-0.0010	-0.0010	1.2562	1.2462	1.3076	1.3070
	1.51	-0.0014	-0.0011	-0.0011	-0.0010	-0.0010	1.2615	1.2544	1.2962	1.2961
0.8	-1.00	0.0012	0.0011	0.0011	0.0010	0.0010	1.0552	1.0530	1.0890	1.0888
	-0.75	0.0014	0.0013	0.0013	0.0014	0.0014	1.0662	1.0643	1.0633	1.0632
	-0.62	0.0015	0.0014	0.0014	0.0013	0.0013	1.0463	1.0442	1.0618	1.0616
	-0.30	0.0010	0.0011	0.0011	0.0011	0.0011	1.0608	1.0595	1.0818	1.0817
	0.30	-0.0010	-0.0011	-0.0011	-0.0013	-0.0013	1.0565	1.0544	1.0686	1.0686
	0.47	-0.0017	-0.0015	-0.0015	-0.0014	-0.0014	1.0560	1.0534	1.0704	1.0703
	0.85	-0.0014	-0.0011	-0.0011	-0.0012	-0.0012	1.0716	1.0698	1.0780	1.0779
	1.51	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	1.0494	1.0475	1.0745	1.0744